# Water and Development

## Part 3c: Groundwater Movement

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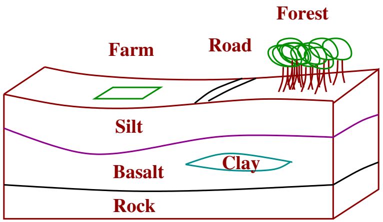
# Recap




Region has many features, both above and below the ground, which impact water balance.

- surface features: infiltration.
- underground features: accumulation and movement of groundwater.

Soil has many parameters related to water:

- specific yield $S_y$: volume fraction of water which is available.
- Conductivity K: ability to move water.

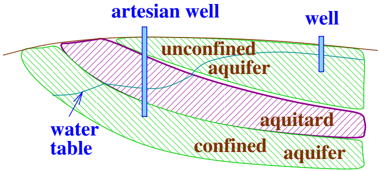
## Aquifer

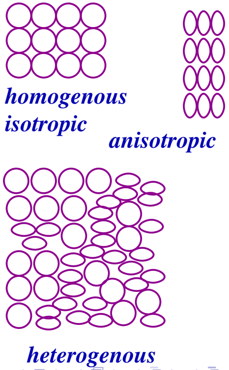
An aquifer is an underground soil-strata which allows the storage and movement of water.

- $K > 5m/d, S_y > 0.1$
- Coarse silts and sands.

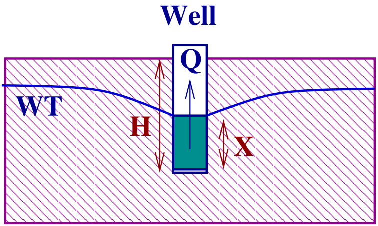
# Aquifers

- aquitards: Soils of poor conductivity.
- Unconfined aquifer: accessible from the surface.
- Confined or partially confined: access blocked or limited.
- Aquifer thickness: The depth to which the aquifer extends.
- Heterogenity, Isotropy directionality and change.
- The water table itself may cross many layers.

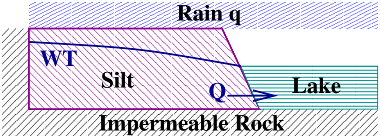


# Our Objective: Real life scenarios



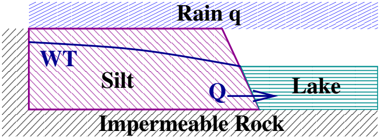

A well is 10m ($H$) deep and $8m$ in diameter and is situated in a farm. The farmer would like to withdraw $Q$ liters/day. Please advice if this is sustainable.




## A lake and its watershed

- Rainfall rate $q$.
- All terrain data is known.
- What is the discharge Q from the banks into the lake?
- What is the water-table WT in the terrain?

# Flows



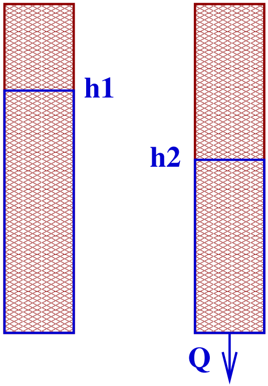

All these questions are about movement of groundwater.

- How much groundwater exists in the ground. Settled through $S_y$ but water table not determined.
- At what rate can groundwater be extracted? Conductivity
- How does the water table interact with the movement of ground water. The hydro-geologic head (or simply total head) and Darcy's law.

# In the lab

- Porosity: The volume fraction of void to solid in dried sample.
- Saturation: When these voids are fully filled with water.

Specific Yield $S_y$: the ration of the colume of water that drains from a rock owing to gravity, to the total rock volume.

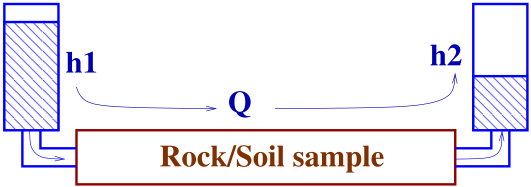



- $h_1, h_2$ resp., are the heights of the saturated layer.
- $Q$ is the volume of the water discharged to reach $h_2$ from $h_1$.
- $S_y = \frac{Q}{(h_1 - h_2)A}$

Caution: rock above $h_i$ is wet, but unsaturated.

**What is the rate of flow?**

# Hydraulic Conductivity




- $h_1$ and $h_2$ are the heights of the water column.
- $Q$ is in cu.m./sec, is the rate of flow.

## Darcy's law

There is a constant $K$ (depending just on the material) so that

$$Q = KA(h_1 - h_2)/L$$

- $Q$ is in cu.m/s
- $L$ is the length of the pipe and $A$ its cross-section area.

# Darcy' law

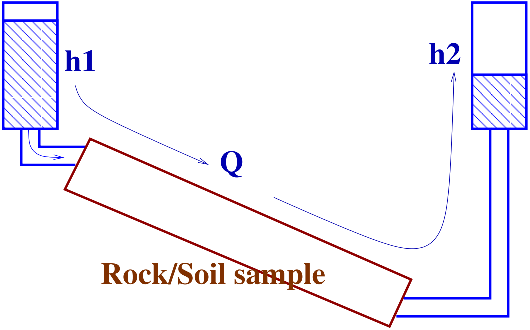
- The first law on the motion of ground-water
- Conductivity K: is an attribute of the substance.
- Dimension of K: is meter/second.

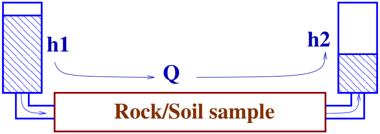
| Material | K in m/d |
|---|---|
| Clay | $10^{-7} - 10^{-4}$ |
| Silts | $10^{-4} - 10^{-2}$ |
| Fine Sands | $10^{-3} - 10^{-1}$ |
| Gravels | $1 - 10$ |

source: Fetter

- Note that Darcy's law *almost* gives us *water particle velocities*.
- WARNING: Only saturated and slow moving flows.
- Typical velocities: few cm a day to few meters a day.
- $K$ actually depends on both the rock/soil and the fluid (e.g., water, oil) which we skip.

# What happens when its sloping?



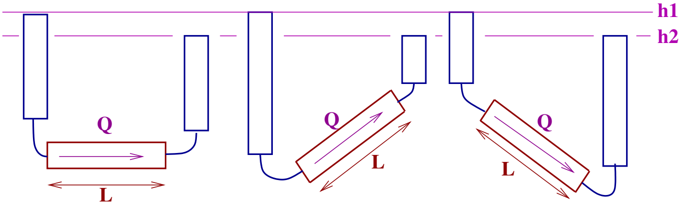




The flow is unchanged as long as the heights of the water columns at the ends is unchanged.

# The General Darcy

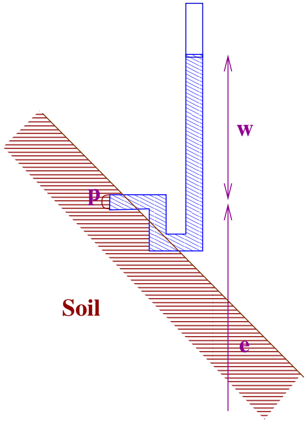
Darcy's observation is that the flow *does not change* even if we vary the angle of inclination provided:

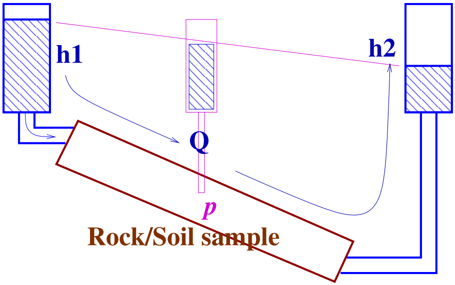
- The length of the rock-sample is not changed.
- The difference in the heads at the ends remains the same.




- This is remarkable in its similarity to ordinary fluid flow.
- It will also lead us to the gradient form of the ground-water differential equation.

# The total head



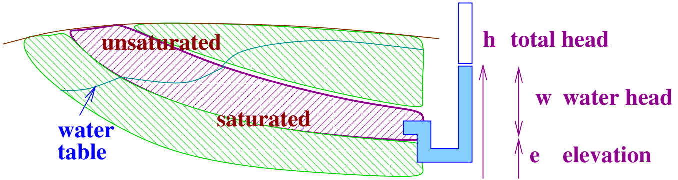




Total head=elevation+hydraulic head
$h(p) = e(p) + w(p)$
The total head varies uniformly within the length of the sample.

# The basic structure

- The ground is itself divided into two parts, the saturated and the unsaturated.
- Moisture equals porosity in the saturated, but diminishes rapidly as we go up.

- The total head is a sum of the water-head and the elevation
- $w \geq 0$ iff the point is saturated.
- The water table is precisely when $w = 0$.



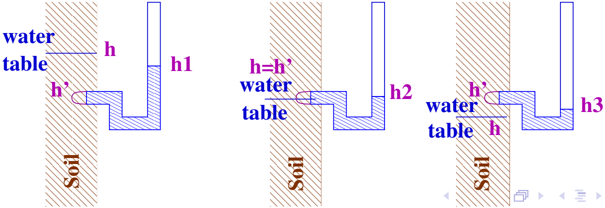
# General Head

- The Piezometer: is a water column with a porous end, and is iused to measure the *piezometric head* at any point in the soil.
- Let h=ht. of water table and h' be the point at which the piezometer is inserted. Let $h_i$ be the readings.

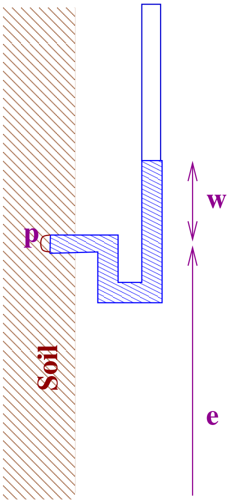
(i) If $h' < h$ then $h' < h1$.

(ii) If $h' = h$ then $h' = h2$.

(iii) If $h' > h$ then $h' > h3$.



# Total Head




- The total head $h(p)$ is the sum of the hydrological head $w(p)$ and the elevation $e(p)$.

$$h(p) = e(p) + w(p)$$

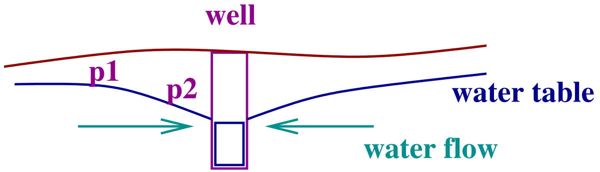
- $w(p) > 0$ iff the point $p$ is saturated.
- $w(p) = 0$ iff $p$ is on the water table.
- $w(p) < 0$ iff $p$ is unsaturated.

## Darcy's Law

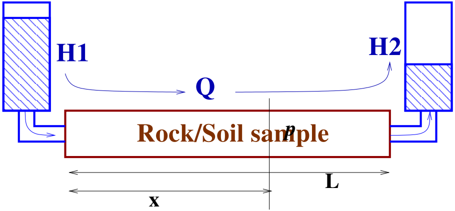
- Water moves from higher total head to lower total head.
- Only applies to the saturated part.
- In the unsaturated part, we assume flows are of significantly smaller values.

# Well Recharge

- Let $p_1$ and $p_2$ be points on the water table.
- Clearly $h(p_i) = e(p_i)$ since $w(p_i) = 0$.
- Thus $h(p_1) > h(p_2)$ and groundwater flows from $p_1$ to $p_2$.
- A well from which water is drawn causes a *dip* in the water table, called the *draw-down cone*.
- This cone causes the well to recharge. The strength of the recharge is given by the angle of attack.
- If the water-table falls below the well-bottom then recharge stops.



# Intermediate heads




- Head at point $p$ at a distance $x$ is say $h(x)$.
- We have by Darcy:

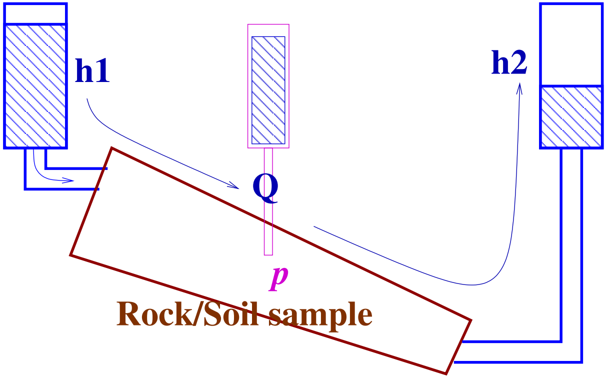
$$Q = \frac{K \cdot (H_1 - h(x)) \cdot A}{x} = \frac{K \cdot (h(x) - H_2) \cdot A}{L - x}$$

- This gives:

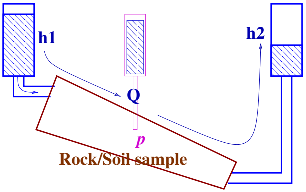
$$h(x) = \frac{(L - x) \cdot H_1 + x \cdot H_2}{L}$$

- Thus $h$ varies uniformly from left to right, or from top to bottom.

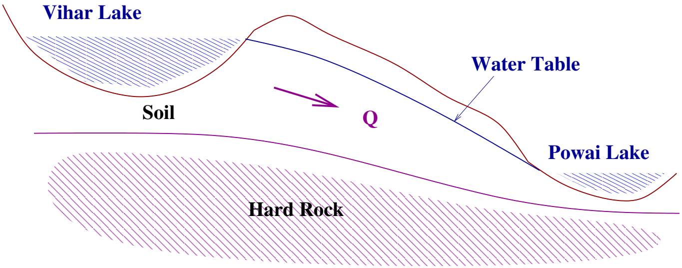
# What if the thickness changes



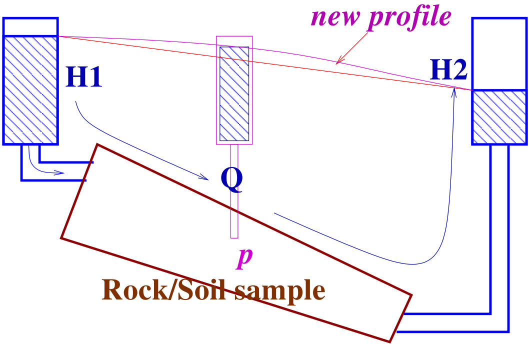
# Not entirely fictitious




**Varying soil thickness!**

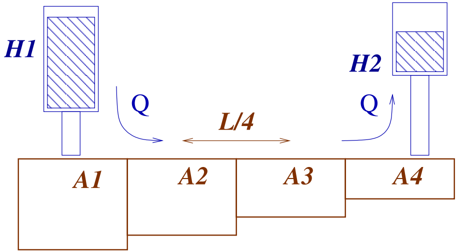


# Coming back: A calculation



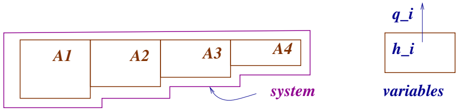

- Notice the change in thickness. Notice that if the thickness is large, the drop is the head is small.
- $\Delta H_1 * KA_1/L = Q = \Delta H_2 * KA_2/L$

# Solving the narrowing pipe 1: Domain Decomposition




- Approximate the system in terms of simpler cells.
- For each cell, associate the internal variable, head $h_i$, and the external variable $q_i$, the external flow coming into the cell.
- Write Darcy's law and conservation law for each cell.

# Posing the narrowing pipe




- Approximate the system in terms of simpler cells.
- For each cell, associate the internal variable, head $h_i$, and the external variable $q_i$, the external flow coming into the cell.
- Write Darcy's law (Note that last equation is superfluous).

$$
\begin{aligned}
q_1 = -q = -q_4, q_2 = q_3 &= 0 \\
h_1 = H_1, h_4 &= H_2 \\
(h_2 - H_1)KA_2/\ell &= -q \\
(H_1 - h_2)KA_2/\ell + (h_3 - h_2)KA_3/\ell &= 0 \\
(h_2 - h_3)KA_3/\ell + (H_4 - h_3)KA_4/\ell &= 0 \\
(h_3 - H_4)KA_4/\ell &= q
\end{aligned}
$$

# Solving

Lets put $A_2 = 4, A_3 = 3, A_2 = 2$ and $K/\ell = \alpha$ to get:

$$\begin{aligned} (h_2 - H_1)4\alpha &= -q \\ (H_1 - h_2)4\alpha + (h_3 - h_2)3\alpha &= 0 \\ (h_2 - h_3)3\alpha + (H_4 - h_3)2\alpha &= 0 \end{aligned}$$

We get:

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 4H_1 \\ 2H_4 \end{bmatrix}$$

$$\frac{1}{26} \cdot \begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 4H_1 \\ 2H_4 \end{bmatrix} = \begin{bmatrix} h_2 \\ h_3 \end{bmatrix}$$

Lets put $H_1 = 20$ and $H_4 = 10$, to get $h_2 = 17.7$ and $h_3 = 14.6$. Now solve for $q$.

Indicates a general method: Domain Decomposition. Break into $n$ pieces and improve accuracy.

# Analytically too...

- Let the cross-section be $A(x)$ at a distance$x$ from the left and let the head be $h(x)$. Whence we have:

$$A(x) = \frac{2 \cdot x + 5 \cdot (L - x)}{L}$$

- We also have:

$$Q = \frac{(h(x + \Delta x) - h(x)) \cdot A(x) K}{\Delta x}$$

- In other words, we have the following equation with the conditions that $h(0) = H_1$ and $h(L) = H_2$.

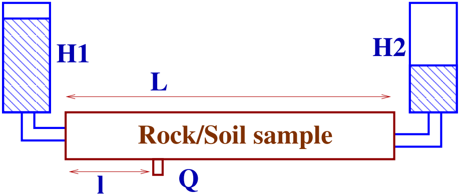
$$\frac{dh}{dx} = \frac{Q}{K \cdot A(x)}$$

- This may be solved and compared with the discrete version.
- However, most problems *cannot* be solved analytically.

# Another problem

A horizontal pipe of length $L$ and cross-section $A$ is held at heads $H_1$ and $H_2$ at its ends. At a distance $l$ from the left end, a tap is made to the pipe and a pump is fitted from which $Q$ cu.m./s is extracted. The conductivity of the soil is $K$.

Find the head at the tapping point. What fraction of $Q$ is coming from the left and what from the right? Can $Q$ be increased without limits? Can you identify a real-life situation?



# Analytic solution

- Let head at point $p$ at distance $\ell$ be $h$. We have

$$\frac{(H_1 - h)AK}{\ell} - \frac{(h - H_2)KA}{L - \ell} = Q$$

- In other words

$$\frac{H_1}{\ell} + \frac{H_2}{L - \ell} - \frac{Q}{KA} = (\frac{1}{\ell} + \frac{1}{L - \ell}) \cdot h$$

- Finally, we have:

$$h = \frac{H_1(L - \ell) + H_2\ell}{L} - \frac{Q \cdot \ell \cdot (L - \ell)}{KAL}$$

- Many interesting things about this expression.

# More details

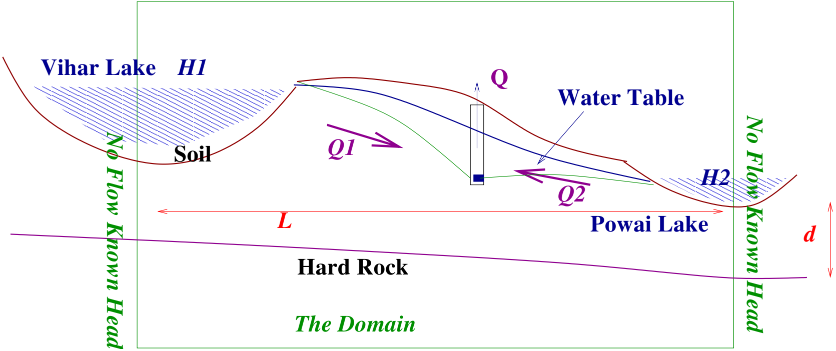
What part comes from the $H_1$ and the $H_2$ side?

$$Q_1 = (H_1 - H_2)\frac{KA}{L} + Q\frac{L-\ell}{L}$$

$$Q_2 = (H_2 - H_1)\frac{KA}{L} + Q\frac{\ell}{L}$$

- We see that for $H_1 > H_2$, for small $Q$, $Q_2 < 0$, i.e., the tap merely diverts some of the flow going from $H_1$ to $H_2$.
- Only when $Q$ is so large that $h$ drops below $H_2$, that $H_2$ is giving some water to the tap.

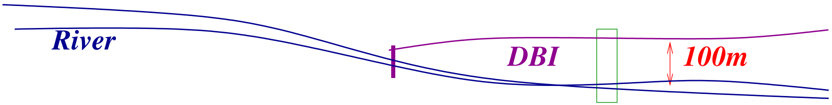
How does this help us in real life?




- If $L >> d >> H_1 - H_2$ then $A$ is roughly constant and the drop in heads is small as compared to the depth of the aquifer.
- The deep aquifer assumption.
- The domain with known *boundary* conditions.
- Such systems are easily solvable.

# DBI system

A DBI breaks off from a river and maintains an average distance of 100m from the river. Moreover, it is roughly 5m higher than the river. If at the entry point, the flow is 100*lps*, how much area can the DBI benefit?



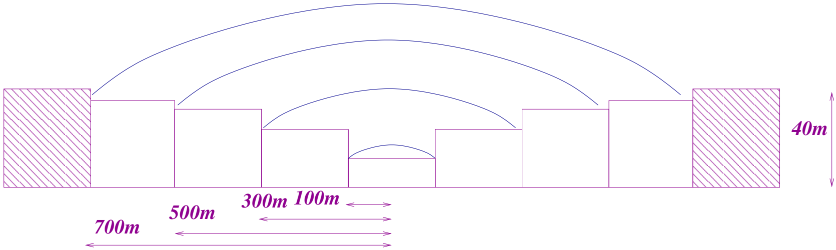

- Let us assume that aquifer thickness is $20m$ and $K = 1m/d$.
- Consider a domain of thickness 1m crossing the canal and the river.
- The flow from the canal to the river will be $Q = 5 \cdot 20/100 = 1$, i.e., 1 cubic meter per day flows from 1m of canal.
- The canal has 100lps=8600 cu.m./s.
- Thus by 8km, the flow in the canal will be reduced to zero.
- How much is the ET for this area?

# The Approximate Well

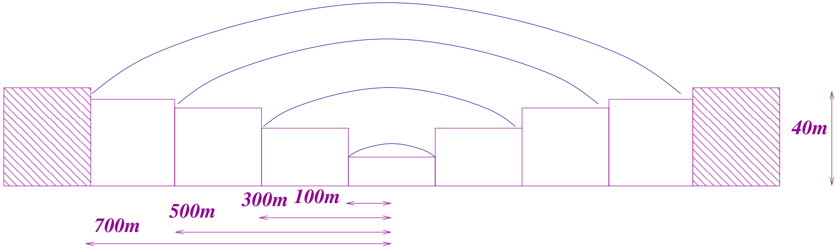
A well of depth $20m$ is being planned to irrigate a land of area 1 hectare. The ambient water table is $5m$ below ground water and the conductivity is $10m/d$. The aquifer depth is $45m$. Will the well provide the water? Let us assume that the bottom of the aquifer is a elevation 0.

We partition the domain into concentric rings, each of width $200m$.

We assume that the impact of the well is zero after $800m$.



# The Approximate Well




- We have the variables $h(800), h(600), h(400), h(200)$ and $h(0)$, where the final variable is the head at the center, i.e., at the well.
- We also have that *far away*, $h(800) = 45 - 5 = 40$

# Computing ...

- Lets assume $Q = 50 cu.m./day$.
- If we know $h(r+\Delta)$, then we may write:

$$\frac{K \cdot (2\pi r \times h(r)) \times (h(r+\Delta) - h(r))}{\Delta} = Q$$

- Takin $\pi = 25/8$, we have:

$$\frac{(500/8) \cdot h(r)) \times (h(r+200) - h(r))}{200} = 50$$

- This is a bit inconvenient, since it is quadratic in $h(r)$. Let us approximate $h(r)$ by $h(r+\Delta)$ and denote the difference by $\Delta h$ to get:

$$h(r+200)) \times \Delta h = 160$$

# Finally...

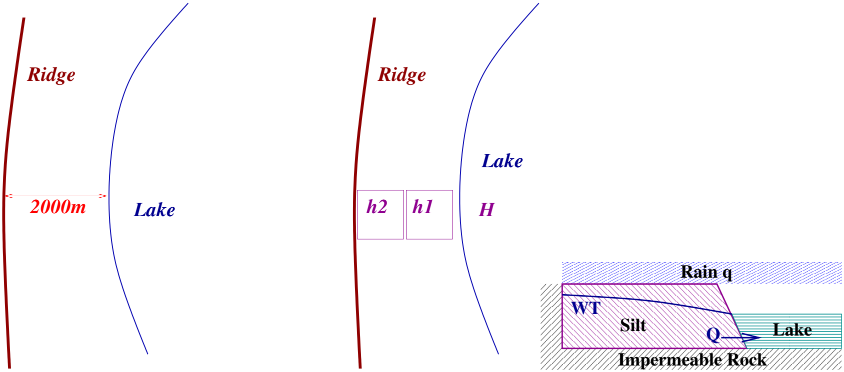
- Thus, we get $h(800) \cdot \Delta h = 160$, t oegt $\delta h = 4$ and thus $h(600) = 36$.
- We have $36 \times |deltah = 160$, to get $\Delta h = 4.5$ and $h(400) = 31.5$.
- $\Delta h$ now becomes 5.1 and $h(200) = 26.4$.
- Finally, we have $\Delta h = 6.1$ with $h(0) = 20.3$.

Thus, the well must operate roughly $5 + (40 - 20.3) = 24.7m$ below ground level. The chosen depth (of $20m$) of the well is insufficient.

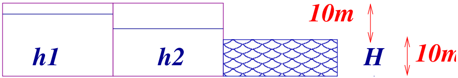
- Examine what happens when $K$ is changed.
- What happens when the *more accurate* quadratic is solved.
- There is an analytic model as well!
- What happens if there is a neighboring well?

# Lake and Ridge

A lake has a ridge on one side, $2km$ away. During the monsoon, it rains about $10mm$ per day with an infiltration of 2%. What is the height of the water table between the lake and the ridge, in the steady state. Assume suitable soil thickness.



# A simple model




- Divide the domain into two squares , $1km \times 1km$, with heads $h_1$ and $h_2$. Thus the net infiltration in each cell 200 cu.m./day.
- *Assuming all flows are underground*, we have:

$$\frac{(h_2 - 10) \cdot K \cdot 1000 \cdot 10}{500} = 400$$

- This gives us $(h_2 - 10) \cdot K = 20$. If $K = 5$, we see $h_2 = 14$.
- The next equation gives $h_1 = 17$.

$$\frac{(h_1 - 14) \cdot 5 \cdot 1000 \cdot 14}{1000} = 200$$

- What would happen if the infiltration were greater? Or $K$ were lower?

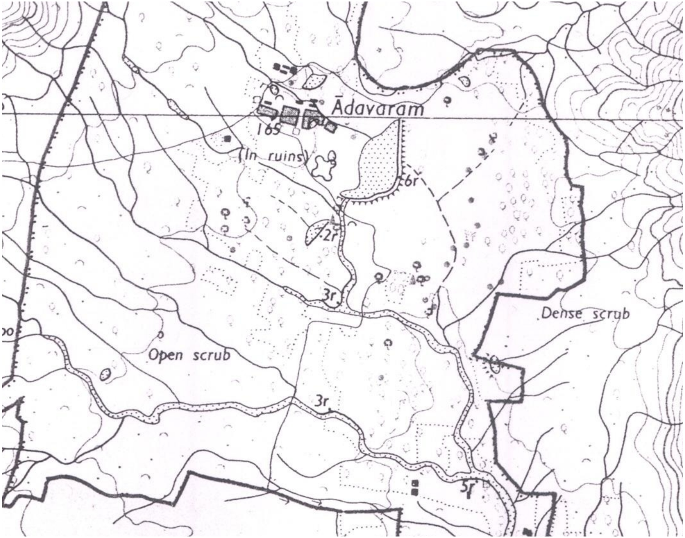
# What happens when it stops raining?

- We have | 17 | 14 | 10 | as the head values in the three cells.
- Thus, we still have a flow of 400 cu.m. out of cell 2 and 200 cu.m. out of cell 1.
- This corresponds to 0.04 and 0.02 mm resp.
- Assuming $S_y = 0.02$, we have $2mm$ and $1mm$ drop on the first day.
- We next update $h_2$ and $h_1$.

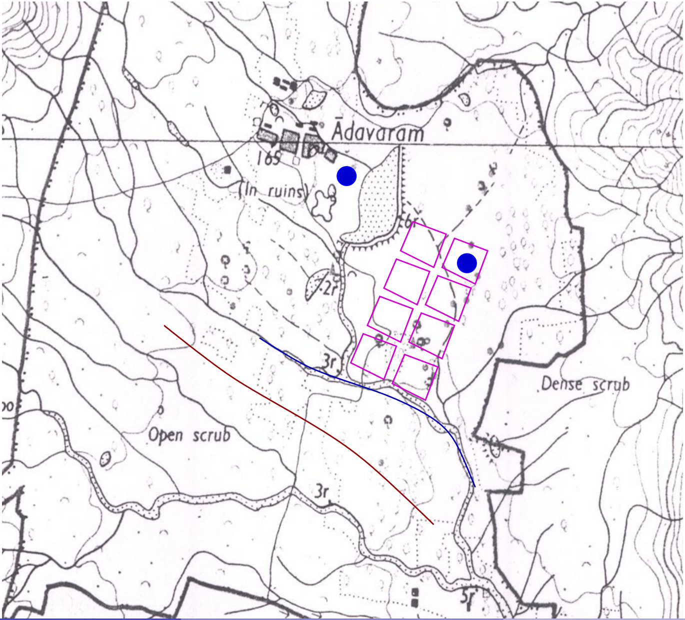
This outlines the dynamic situation as well.

Changes in head values due to infiltration generally outweigh those due to movements.

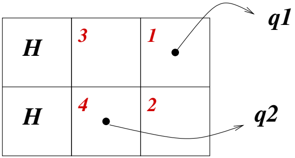
# Real life...



# Adjacent farms



# The model




- 6 squares, each side 1 km. The left-most adjoin the river, at a fixed head $H$.
- Two wells in two diagonal squares.
- What is the impact of the wells on each other?

- Let us write the conservation equations.
- Denote $KA/L = \alpha$ and $Q_i = q_i/\alpha$.
- *Deep aquifer*. $A$ does not depend on $h$.

$$\begin{aligned} (h_3 - h_1) + (h_2 - h_1) &= Q_1 \\ (h_1 - h_2) + (h_4 - h_2) &= 0 \\ (H - h_3) + (h_4 - h_3) + (h_1 - h_3) &= 0 \\ (H - h_4) + (h_3 - h_4) + (h_2 - h_4) &= Q_2 \end{aligned}$$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} Q_1 \\ 0 \\ -H \\ Q_2 - H \end{bmatrix}$$

Let us denote this equation as $Gh = b$.

# Whence...

Thus $h = G^{-1}b$ where $G$ is actually:

$$\frac{1}{11}\begin{bmatrix} -13 & -9 & -6 & -5 \\ -9 & -13 & -5 & -6 \\ -6 & -5 & -7 & -4 \\ -5 & -6 & -4 & -7 \end{bmatrix}$$

Let us assume that $H = 2$.

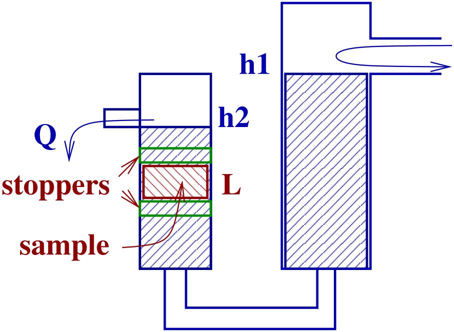
- If $Q_1 = Q_2 = 0$, then $h_i = 2$ for all $h_i$ as expected.
- If $Q_1 = 1$ and $Q_2 = 0$, we have $h = [0.82, 1.18, 1.45, 1.55]$

- More interesting, if $Q_1 = 0$ and $Q_2 = 1$, we have

| | | |
|---|---|---|
| 2 | 1.64 | 1.55 |
| 2 | 1.36 | 1.45 |

- Note that the heads depend as a linear combination of $Q_1$ and $Q_2$ and that decides the well-interference. For each unit of extraction at $Q_2$, there is a loss of $0.45m$ of head at cell 1.
- If $K$, the conductivity increases then $\alpha$ increases and the head-loss decreases.

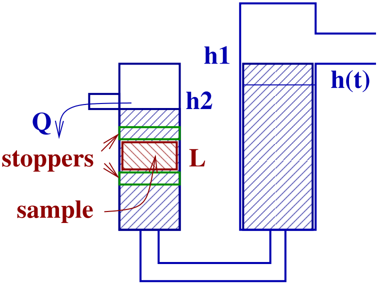
# Measuring K: fixed heads




- The head difference is maintained at $h_1 - h_2$.
- The sample is held by two permeeable stoppers.
- The sample thickness is $L$ and cross-section $A$.
- The system is at steady state and the outflow $Q$ is measured.

$$K = \frac{QL}{A \cdot (h_1 - h_2)}$$

# Measuring K: varying heads




- Start with height $h(0) = h_1$ and stop after time $T$ and at height $h(T)$.
- Let cross-section of both tubes be $A$.
- Let $Q$ be the total water discharged.

- We have $Q = KA(h(t) - h_2)/L$, whence we have:

$$dh/dt = -K(h(t) - h_2)/L \qquad h(0) = h_1$$

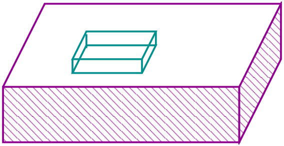
- $h(t) = (h_1 - h_2)e^{-Kt/L} + h_2$ whence we have:
- $K = L \log[(h_1 - h_2)/(h(T) - h_2)]/T$

# Farm-pond

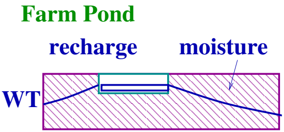
A farmer is considering a farmpond of size $10m \times 30m \times 2m$, of about Rs. 10,000 in direct and indirect costs. The objectives are:

- Recharge for better moisture for *rabi*.
- Use for paddy crop in dry spells.

Please advise.



- A real-life techno-economic problem.
- Mainly unsaturated flows (moisture) and transient analysis.
- Crop related information: wilt-points.
- Evaporation-Transpiration rates and Infiltration.
- Monsoon behaviour.



# Thanks