Stabilizer Limits and Alignment - Lie Algebraic Methods for the Orbit Closure Problem.

Bharat Adsul, Milind Sohoni - IIT, Bombay
K V Subrahmanyam - CMI, Chennai

Key Reference: *Orbit Closures, Stabilizer Limits and Intermediate G-varieties*, arxiv 2309.15816 [1]

24th February 2025

Outline

- Problem definition and motivation
- Stabilizer limits - Main result, some examples. The genericity "obstacle" and Plans A and B.
- Plan A (Alignment). Examples and a "genericity" result
- Consequences - An entry point for combinatorial analysis?
- Classical cases
  - Instability, Kempf optimal one-PS
  - Unstable and stable points - alignment and tangent-vector orbits
- Plan B - The Pictures
- Going ahead - Lie algebraic evidence to algebraic geometry.
  - Co-dimension 1 varieties in $\overline{O(det_3)}$.

Notation

- X over $\mathbb{C}$ and $dim(X) = n$. $G \subseteq GL(X)$, connected reductive *algebraic* group over $\mathbb{C}$. Typically $G = GL(X) = GL_n(\mathbb{C})$.
- $\rho : GL(X) \to GL(V)$, representation such that the center $Z = \{tI | t \in \mathbb{C}^*\}$ acts as $\rho(tI)(v) = t^d v$ for a fixed d. Moreover, $Z \subseteq G$. Think $V = Sym^d(X^*)$.
- $y \in V$. *Orbit* of y , $O(y) := \{g \cdot y | g \in G\}$.
- $O(y)$ need not be closed, it is constructible.
- $\overline{O(y)}$, *orbit closure* of y - *Zariski* topology or *Euclidean topology*. $\overline{O(y)}$ is a cone and its $I(y) \subseteq \mathbb{C}[V]$ is homogeneous.
    - GL_n action on $\mathbb{C}^n$. $\overline{O(v)} = \mathbb{C}^n$, $v \in \mathbb{C}^n, v \neq 0$.
    - GL_n adjoint action on M_n. $\overline{O(J_n)} = \mathcal{N}$, the *nilpotent cone*.

The Question of interest

Question:

- Given $z, y \in V$, is $z \in \overline{O(y)}$? Distinctive stabilizers, G_z, G_y.
- Given $[z], [y] \in \mathbb{P}(V)$, is $[z] \in \overline{O([y])}$?
- MOTIVATION – algebraic complexity theory.

Conversely

- Given 1-PS $\lambda(t) \subseteq G$ and the action:

$$\lambda(t)y = t^d z + \ldots + t^D y_D$$

What connects $K = G_y$ and $H = G_z$?

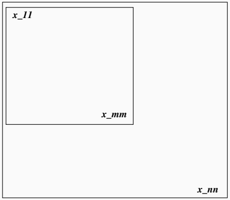
- By applying a suitable power $t^a I$, we have:

$$\lambda'(t)y = t^0 z + \ldots + t^{D'} y_{D'}$$

Thus $z \in \overline{O(y)}$.

The permanent vs. determinant question

$X = \mathbb{C}^{n\times n}$, $V = Sym^n(X)$ and $y = det_n(X)$, $z = x_{nn}^{n-m} perm_m$. Is there a homogenous substitution AX so that the determinant $det_n(AX) = x_{nn}^{n-m} perm_m(X_{mm})$?



Therefore there is a 2-block 1-PS $\lambda_A(t) \subseteq G$ such that:

$$\lambda_A(t)y = z + \sum_{i>0} t^i y_i$$

Note that this puts $z = x_{nn}^{n-m} perm_m \in \overline{O(y)}$ where $y = det_n$.

Question: What is the smallest n which does the job? Note that this does not require stabilizer containment!

Stabilizers

K_n = Stabilizer of $det_n \in Sym^n(X_n)$

What is the stabilizer of det_n in $GL(X)$?

- $X_n \to CX_nD$ such that $C, D \in GL_m$ and $det_n(CD) = 1$ and $X \to X^T$.
- $K = G_y$ is reductive, $dim(G_y) = 2n^2 - 1$ and X_n is an irreducible G_y-module.

H_m =Stabilizer of $perm_m \in Sym^m(X_m)$

What is the stabilizer of $z' = perm_m$ in $GL(X_m)$?

- $X_m \to CX_mD$ such that $C, D \in D_m$ and $det_m(CD) = 1$ and $X \to PX^TP'$, with P, P' permutation matrices.
- $G_{z'}$ is reductive, $dim(G_{z'}) = 2m - 1$ and X_m is an irreducible G_{perm_m}-module.

More stabilizers and GCT

$H_{n,m}$ =The stabilizer of the homogenized permanent

$z = x_{nn}^{n-m} perm_m(X_m) \in Sym^n(X_n)$. We may divide $X_n = \overline{X'_m} \oplus \mathbb{C}x_{nn} \oplus X_m \cong X_1 \oplus X_0$. Then $H_{n,m} = G_z \subseteq GL(X)$ in the ordered basis is as below:

$$\left[\begin{array}{c|cc} * & * & * \\ \hline 0 & * & 0 \\ 0 & 0 & g \end{array}\right] \quad \text{with } g \in H_m$$

We also have the limit: $\lambda(t) \cdot y = z + \sum_{i>0} t^i y_i$.

Stabilizers change dramatically under taking limits!

- Both det_n and $perm_m$ are SL-stable (their orbits are closed) and *determined* by their stabilizers in their respective spaces.
- Stabilizer data enough to determine containment of $z \in \overline{O(y)}$.

GCT and Representations as Obstructions

- Let $Y = \overline{O(y)}$ and $Z = \overline{O(z)}$, and $\mathbb{C}[Y] = \sum_\mu d_\mu V_\mu$ and $\mathbb{C}[Z] = \sum_\mu p_\mu V_\mu$ be their coordinate rings as G-modules.
- Stability of $det_n, perm_m$ and Peter-Weyl determine exactly which G-modules V_u appear in $\mathbb{C}[Y]$ and $\mathbb{C}[Z]$.
- $Z \subseteq Y \Rightarrow \mathbb{C}[Y] \twoheadrightarrow \mathbb{C}[Z]$ and thus $d_\mu \geq p_\mu$ for all μ.

GCT-II Conjecture

If $z \notin Y$ then there is a μ such that $p_\mu > 0$ and $d_\mu = 0$.

And its failure...

All V_μ which appear in $\mathbb{C}[Z]$, or for that matter, for the coordinate ring $\mathbb{C}[W]$ of the orbit closure $\overline{O(w)}$ of any homogenized form w, appear in $\mathbb{C}[Y]$.

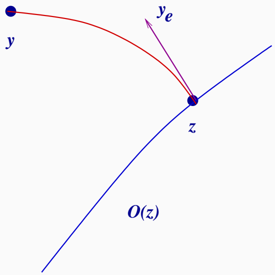
So the numbers do matter.

Our work - more geometric

We begin with:

$$y(t) = \lambda(t).y = y_d t^d + y_e t^e + \sum_{i=e+1}^{D} y_i t^i$$

with $z = y_d$. We call y_e as the tangent of approach.
We use the notation $y \xrightarrow{\lambda} z$.



Transversality Assumption. Vector space spanned by $y_e, \ldots, y_D$ intersects $T_g O(g)$ trivially. Let $K = G_y$ and $H = G_z$.
Let $\mathcal{G} = Lie(G)$ and
$\mathcal{K} = Lie(K), \mathcal{H} = Lie(H) \subseteq \mathcal{G}$.

Question

How do we connect $\mathcal{K}$ with $\mathcal{H}$ using λ?

- We have the usual action of λ on V and the weight space decomposition $V = \oplus V_i$.
- $\lambda(t)$ also acts on $\mathcal{G}$ by conjugation and thus we have $\mathcal{G} = \oplus \mathcal{G}_i$.
- For any $v \in V, v = \sum_i v_i$, let the **leading term** $\widehat{v}^{\lambda}$ or simply $\hat{v}$ be v_j where $v_j \neq 0$ and $v_i = 0$ for all $i < j$. Similarly, we define $\widehat{\mathfrak{g}}^{\lambda}$ or simply $\hat{\mathfrak{g}}$ for any $\mathfrak{g} \in \mathcal{G}$.

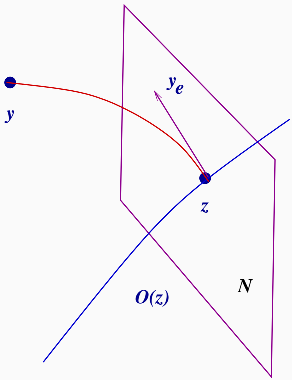
Basic result: For any $\mathfrak{g} \in \mathcal{G}$ and $v \in V$:

$\lambda(t)(\mathfrak{g}v) = (\lambda(t)\mathfrak{g}\lambda^{-1}(t))(\lambda(t)v) = \mathfrak{g}(t)v(t)$. Thus either $\hat{\mathfrak{g}}\hat{v} = 0$ or $\widehat{\mathfrak{g}v} = \hat{\mathfrak{g}}\hat{v}$ and $deg(\mathfrak{g}v) = deg(v) + deg(\mathfrak{g})$.

Proposition

Let $\mathcal{K}$ be a Lie subalgebra of $\mathcal{G}$ and $N \subseteq V$ a $\mathcal{K}$-module. Then
(i) $\hat{\mathcal{K}}$ is a graded Lie subalgebra of $\mathcal{G}$, and $dim_{\mathbb{C}}(\hat{\mathcal{K}}) = dim_{\mathbb{C}}(\mathcal{K})$,
(ii) $\hat{N} \subseteq V$ is a $\hat{\mathcal{K}}$-module with $dim_{\mathbb{C}}\hat{N} = dim_{\mathbb{C}}N$.

The $\overline{N}$-action

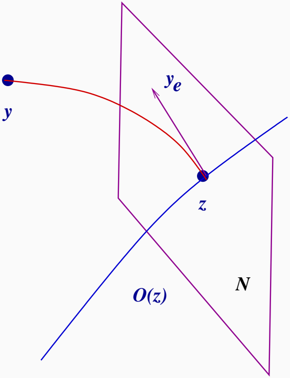


The condition:

$$\lambda(t) \cdot y = t^d z + t^e y_e + \ldots + t^D y_D$$

implies that $\hat{y} = z$. In other words
$\hat{y}^\lambda = z \Leftrightarrow y \xrightarrow{\lambda} z$

- Let $T_z(O(z)) \subseteq V$ be the tangent space of $O(z)$ at z and N be a complement.
- $T_z \subseteq V$ is an $\mathcal{H}$-module and so is $\overline{N} = V/T_z$.
- $\overline{y_e} \in \overline{N}$ and $\mathcal{H}_{\overline{y_e}}$ its stabilizer.



Theorem (ASS)

Let $y \overset{\lambda}{\to} z$ with stabilizers Lie algebras $\mathcal{K}, \mathcal{H}$ as above. Let $\overline{N}$ be the the quotient $V/T_zO(z)$ and $\overline{y_e} \in \overline{N}$. Then we have $\hat{\mathcal{K}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$.

Proof: (Assume $e = d+1$). If $\mathfrak{k} \in \mathcal{K}$, then we have: $\mathfrak{k} \cdot y = (\lambda(t)\mathfrak{k}\lambda(t)^{-1}) \cdot (\lambda(t)y) = \mathfrak{k}(t) \cdot y(t) = 0$.

If $\mathfrak{k}(t) = \sum_{i \geq i_0} t^i \mathfrak{k}_i$ and $y(t) = \sum_{j \geq d} t^j y_j$ then we have $\hat{\mathfrak{k}} = \mathfrak{k}_{i_0}$ and :

$$\begin{aligned} \hat{\mathfrak{k}} y_d &= 0 \\ \hat{\mathfrak{k}} y_e + \mathfrak{k}_{i_0+1} y_d &= 0 \end{aligned}$$

Permanent vs. Determinant

Therefore...

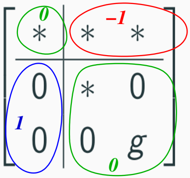
If $z = x_{nn}^{n-m} perm_m = det_n(AX_n)$, then $z = \widehat{det_n}^{\lambda}$ for a suitable 2-block λ_A. Thus $\hat{\mathcal{K}}_n \subseteq \mathcal{H}_{n,m}$. How does $\hat{\mathcal{K}}_n$ sit inside $\mathcal{H}_{n,m}$?

Recall $X_n = \overline{X'_m} \oplus \mathbb{C}x_{nn} \oplus X_m \cong X_1 \oplus X_0$. Then $H_{n,m}$ is as below (with $g \in H_m$):

$$\left[\begin{array}{c|cc} * & * & * \\ \hline 0 & * & 0 \\ 0 & 0 & g \end{array}\right]$$

Given a $\mathfrak{k} \in \mathcal{K}_n$ with $\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1$.

As per the weights of λ_A, we have:



What if $\mathfrak{k}, \hat{\mathfrak{k}} = \mathfrak{k}_{-1}$ for all $\mathfrak{k}$? Then the stabilizer of det_n will be tucked away from H_m! **Can λ_A be "generic"?**

Measuring Generic-ness

For $\lambda(t)$ be as below, see the weight-spaces:

$$\lambda(t) = \begin{bmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathcal{G} = \left[\begin{array}{c|c|c} \mathcal{G}_0 & \mathcal{G}_{-1} & \mathcal{G}_{-2} \\ \hline \mathcal{G}_1 & \mathcal{G}_0 & \mathcal{G}_{-1} \\ \hline \mathcal{G}_2 & \mathcal{G}_1 & \mathcal{G}_0 \end{array}\right]$$

Thus, for a general λ, $\hat{\mathcal{K}} = \oplus_i \hat{\mathcal{K}}_i$, with $dim(\hat{\mathcal{K}}_i) = k_i$. The vector $\overline{k} = (k_i)$ measures the generic-ness of λ vis a vis $\mathcal{K}$. The more negative the weights, the more generic is λ.

What if, λ_A is completely generic and $\overline{k}$ is as follows:

weight	-1	0	1
dimension	$dim(\mathcal{K}_n)$	0	0

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Can interesting forms be generic limits of det_n?

Like to believe that the answer is NO

Two Questions

Theorem (ASS)

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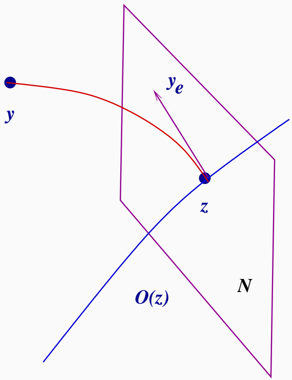
Two questions.

- Plan A. (Alignment) Is there a common semisimple element between $\mathcal{K}$ (or its conjugate) and $\mathcal{H}$.
- Plan B (Lie algebra) Are there intermediate orbits $\overline{O(z)} \subset \overline{O(w)} \subset \overline{O(y)}$ which are simpler? Do the containments $\hat{\mathcal{K}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$ give us a clue?

Note that the containment $\overline{O(z)} \subset \overline{O(w)} \subset \overline{O(y)}$ may happen without the reverse containment of stabilizers. Indeed, is there a sequence of such orbits where each step in simple?



Therefore...

If $z = x_{nn}^{n-m} perm_m = det_n(AX_n)$, then $z = \widehat{det_n}^{\lambda}$ for a suitable 2-block λ_A. Thus $\hat{\mathcal{K}}_n \subseteq \mathcal{H}_{n,m}$.

How does $\hat{\mathcal{K}}_n$ sit inside $\mathcal{H}_{n,m}$?

Alignment

A semisimple element $\mathfrak{s} \in \mathcal{K}$ is called an alignment if it commutes with λ.

Observe: If $\mathfrak{s}$ is an alignment and $\lambda(t)y = t^d z + t^e y_e + \ldots + t^D y_d$ then $\mathfrak{s}(\lambda(t)y) = \lambda(t)\mathfrak{s}y = 0$. Thus $\mathfrak{s}$ stabilizes every y_i and therefore $z = y_d$. **Thus $\mathfrak{s} \in \mathcal{H}$.**

Example: det_3

Let $X = X_3$ be as below and let $det_3(X) \in Sym^3(X)$ be the usual determinant and three 2-block 1-PS with the same 6-3 break:

$$X_3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \lambda_A = \begin{bmatrix} tx_1 & tx_2 & tx_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

$$\lambda_B = \begin{bmatrix} tx_1 & x_2 & x_3 \\ x_4 & tx_5 & x_6 \\ x_7 & x_8 & tx_9 \end{bmatrix} \lambda_C = \begin{bmatrix} x_1 & tx_2 & tx_3 \\ x_4 & x_5 & tx_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

We have the following limits:

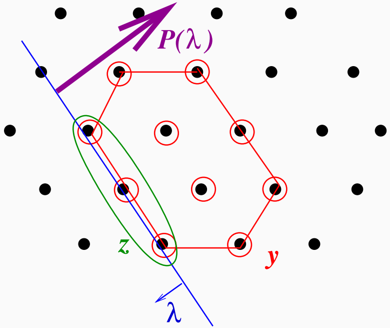
	limit	*degree*	*dim*($\mathcal{H}$)	-1	0	1
$\hat{\mathcal{K}}_A$	det_3	1	16	0	16	0
$\hat{\mathcal{K}}_B$	*derangements*	0	31	12	4	0
$\hat{\mathcal{K}}_C$	$x_1x_5x_9$	0	56	14	2	0

$P(\lambda), U(\lambda)$ generalities

Let $T \supseteq \lambda(t)$ be a maximal torus and $\Xi(V)$, the weight space. Let $\mathcal{T} = Lie(T)$. For any $\mathfrak{t} \in \mathcal{T}$, let $t^{\mathfrak{t}}$ be the 1-PS corresponding to $\mathfrak{t}$. Let us assume that $\lambda'(t)$ is such that $d = 0$, i.e., $y(t) = y_0 + t^1 y_1 + \ldots + t^D y_D$ with $z = y_0$.
Let ℓ be such that $t^{\ell} = \lambda'(t)$. Thus λ' stabilizes z and $\ell \in \mathcal{H}$. Let $\mathcal{G}_i$ be the weight space decomposition of $\mathcal{G}$ w.r.t λ (or λ').

- Recall $P(\lambda) = \{p \in G | \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists}\}$.
- $L(\lambda)$ is precisely elements of $P(\lambda)$ which commute with λ.
- There is a Levi decomposition $P(\lambda) = L(\lambda) \ltimes U(\lambda)$, with $L(\lambda)$ reductive and $U(\lambda)$ unipotent.
- $Lie(P(\lambda)) = \mathcal{P}(\lambda) = \oplus_{i \geq 0} \mathcal{G}_i$, $Lie(U(\lambda)) = \mathcal{U}(\lambda) = \oplus_{i > 0} \mathcal{G}_i$ and $Lie(L(\lambda)) = \mathcal{L}(\lambda) = \mathcal{G}_0$.

The Picture



- We have $V = \oplus_r V_r$, the λ-decomposition and the $P(\lambda)$-space $V_{\geq 0} = \oplus_{i \geq 0} V_i$.
- We also have the $L(\lambda)$-equivariant projection $\pi_i : V \to V_i$, and in particular $\pi_0 : V_{\geq 0} \to V_0$.

Lemma

For any $p \in P(\lambda)$ with $p = us$, where $u \in U(\lambda), s \in L(\lambda)$, we have $\widehat{py}^{\lambda} = sz \in O(z)$.

Theorem: Nilpotency or Alignment

Let $\overline{U}(\lambda) = U(\lambda(t^{-1}))$ be the *opposite* unipotent group and $\overline{\mathcal{U}}(\lambda) = \oplus_{i<0}\mathcal{G}_i$ be its Lie algebra. We then have:

$$\mathcal{G} = \overline{\mathcal{U}}(\lambda) \oplus \mathcal{L}(\lambda) \oplus \mathcal{U}(\lambda)$$

Proposition

Either there is a $\mathfrak{k}$ such that $deg(\hat{\mathfrak{k}}) \geq 0$ or $\hat{\mathcal{K}} \subseteq \overline{\mathcal{U}}(\lambda)$ and is nilpotent and there is a $\mathfrak{u} \in \overline{\mathcal{U}}(\lambda)$ such that $[\mathfrak{u}, \hat{\mathcal{K}}] = 0$. For λ_A in Valiant's construction, $\mathfrak{u} \in \mathcal{H} - \hat{\mathcal{K}}$. The normalizer!

Theorem

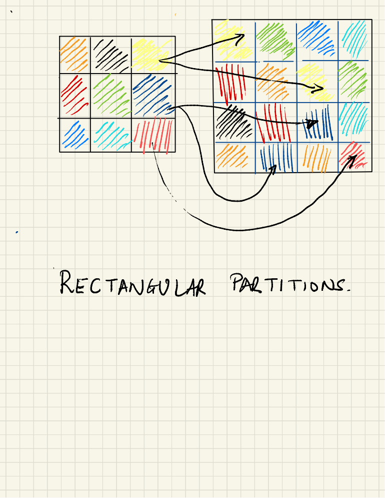
Let y, z, λ be as above and $\mathcal{H} = \mathcal{G}_z$ and $\mathcal{K} = \mathcal{G}_y$. Then either (i) there is a $u \in U(\lambda)$ such that $\widehat{uy}^{\lambda} = z$ and a semisimple $\mathfrak{s} \in \mathcal{G}_{uy}$ which commutes with λ, OR (ii) $\hat{\mathcal{K}} \subseteq \mathcal{H}$ is a nilpotent Lie alegbra.

Consequences of Alignment

If there is an alignment $\mathfrak{s} \in \mathcal{K}_n$, the stabilizer of det_n and $x_{nn}^{n-m} perm_m$ via λ_A for some A. Then there is a 1-PS $u^{\mathfrak{s}} = \mu(u)$ such that the weight spaces of $X_m \dot{\cup} \{x_{nn}\}$ and X_n are linked by A.

- Variables $\{x_{11}, \ldots, x_{mm}\} \cup \{x_{nn}\}$ of $x_{nn}^{n-m} perm_m$ get partitioned into rectangles, and variables $\{x_{11}, \ldots, x_{nn}\}$ of the determinant get partitioned into rectangles.
- Each rectangle corresponds to the weight spaces w.r.t μ.
- The map A puts the permanent variables into the corresponding rectangles of the determinant.
- For both the permanent and the determinant, these rectangular spaces are also linear subspaces within their respective hypersurfaces.

Entry point for combinatorial analysis?



Alignment in Grenet's construction

- Grenet's implementation of the permanent is also via rectangular partitions

$$\begin{bmatrix} 0 & 0 & 0 & 0 & x_{33} & x_{32} & x_{31} \\ x_{11} & x_{77} & 0 & 0 & 0 & 0 & 0 \\ x_{12} & 0 & x_{77} & 0 & 0 & 0 & 0 \\ x_{13} & 0 & 0 & x_{77} & 0 & 0 & 0 \\ 0 & x_{22} & x_{21} & 0 & x_{77} & 0 & 0 \\ 0 & x_{23} & 0 & x_{21} & 0 & x_{77} & 0 \\ 0 & 0 & x_{23} & x_{22} & 0 & 0 & x_{77} \end{bmatrix}$$

- $I = \{1\}\{2\}\{3\}\{7\}$ and $J = \{1,2,3\}\{7\}$ for permanent variables.
- $I = J = \{1\}\{2,3,4\}\{5,6,7\}$ for determinant variables.

Alignment - Relating eigenspaces of stabilizers

The eigenspaces of semi-simple elements of $perm_n$ or $\det_n$ happen to be similar. Moreover, these are linear supspaces of the corresponing hypersurfaces.

Result (Ressayre - Mignon)

If $perm_m$ is obtained as a pull-back of det_n, then $n > m^2/2$. Analysis of the curvature tensor of the hypersurfaces.

Proposition (ASS)

Suppose that, there is a sequence of points $(p_m) \in P_m$ and a function $k(m)$, and the guarantee that the dimension of any linear subspace $L \subseteq P_m$ containing p_m is bounded by $k(m)$. If $perm_m$ is obtained as a pull-back of $det_n(X)$ is $perm_m(W)$. Then $n \geq m^2 - k(m)$.

Conjecture: $k(m) = o(m^2)$.

Classical Case - Unstable and semistable points $G = GL(X)$

Question

How does our analysis apply to classical limits in GIT?

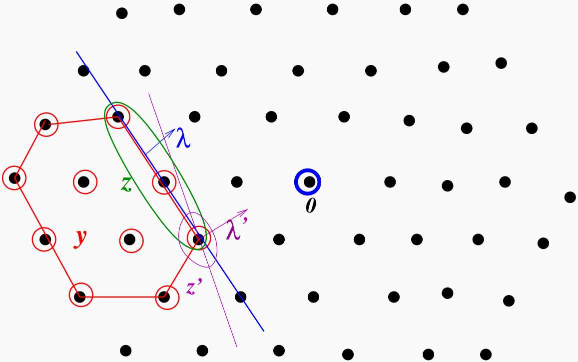
Definition: Instability

Let S be a closed G-invariant subset of V. Then, y is said to be S-unstable if $\overline{O_{SL(X)}(y)}$ intersects S. If $S = \{0\}$, then S-unstable is called unstable. If $S = O(z)$, an $SL(X)$-closed orbit, then y is called semi-stable.

Hilbert-Mumford criterion

y is S-unstable, if and only if there exists $\lambda(t) \in SL(X)$ such that if $\lambda(t)y = t^d y_d + t^e y_e \ldots + t^D y_d$, then $d = 0$ and $y_0 \in S$.

Two cases: (1) y unstable, limit $z = 0$ and tangent y_e (2) y semistable, limit $z \neq 0$ and tangent y_e.



There is a unique optimal λ' (upto conjugation by $u \in U(\lambda')$).

$$\lambda'(t)y = y'_{e'}t^{e'} + \ldots + y_{D'}t^{D'}$$

with $e' > 0$. Moreover, $u\lambda' u^{-1}y = y'_{e'}t^{e'} + \ldots$ so $e'_{e'}$ is well defined. Thus $y \xrightarrow{\lambda'} 0$ and y_e is the tangent, $K = G_y$ and $H = G$.

The unstable case - by Kempf

- The stabilizer subgroup K of G is contained in $P(\lambda')$.
- Let $R \subseteq K$ be a reductive subgroup fixing y. Then there is an optimal $\lambda'' = u\lambda' u^{-1}$, with $u \in U(\lambda')$ which commutes with R. Thus, if K has semisimple elements, then we may choose λ' to commute with a maximal subgroup and alignment holds.
- Since the limit $z = 0$, $\overline{O(z)} = \{0\}$ and $\overline{N} = V$. Hence $H_{\overline{y'_{e'}}} = G_{y'_{e'}}$. If this does not equal K, then $0 \subsetneq \overline{O(y'_{e'})} \subsetneq \overline{O(y)}$, and the required intermediate variety also exists.
- Indeed, if $S = \overline{O(y'_{e'})}$, then S is a cone and y is S-unstable and λ' itself is a witness to it.
- True for general reductive G.

The semi-stable Case 2 - by Kempf and Luna

Let z be a stable point, i.e., $O_{SL(X)}(z)$ be closed. Example: det_n or $perm_m$. Let $H = G_z$ and $K = G_y$.

Let y be S-unstable and λ be Kempf-optimal and so:

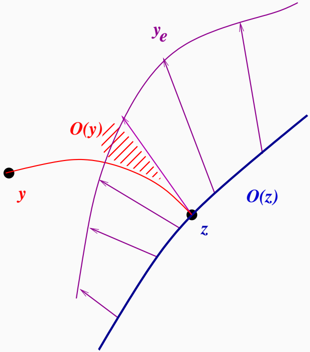
$$\lambda(t)y = z + t^e y_e + \ldots + t^D y_D$$

- By Luna, (i) H is reductive and we may assume that there is an H-module N complementary to $T_z O(z)$.
- We may then assume that $y \in z + N, K \subseteq H$ and $\lambda \subset H$.
- This then reduces to Case 1 with the reductive group H replacing G and N replacing V.
- Thus semisimple elements in K descend to H.
- Consider $O_H(y_e) \subseteq N$ and let $W = \overline{G \times^H O_H(y_e)}$. Then $\overline{O(z)} \subsetneq W \subsetneq \overline{O(y)}$. Thus the intermediate variety condition holds as well.

Plan B - The Pictures - The tangent vector

Lets look at..

The two block case and $\hat{\mathcal{H}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$.



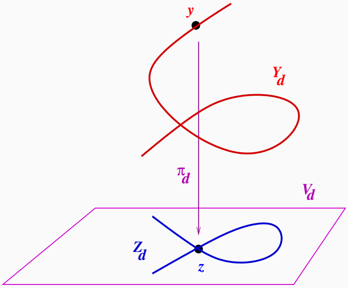
This examines the gap $\hat{\mathcal{K}} \subsetneq \mathcal{H}_{\overline{y_e}}$. Then $dim(O(y))$ in V is greater than $dim(O(\overline{y_e}))$ in $G \times^H \overline{N}$.

So is there...

an element $w \in V$ with stabilizer $\mathcal{H}'$ such that $\widehat{\mathcal{H}'}^{\mu} = \mathcal{H}_{\overline{y_e}}$? Is there an "extension" of y_e into V?

Would indicate $\overline{O(z)} \subsetneq \overline{O(w)} \subsetneq \overline{O(y)}$, help in finding forms simpler than det_n with $x_{nn}^{n-m} perm_m$ as limits.

Plan B - The Pictures - Co-limits



This examines the gap $\mathcal{H}_{\overline{y_e}} \subsetneq \mathcal{H}$.

Let $Y_d = O(y) \cap V_{\geq d}$ and $Z_d = \pi_d(Y_0)$. Note that $y \in Y_d$ and $z \in Z_d$, the space of co-limits of z. Let $Z = \overline{O(Z_d)}$, then $\overline{O(z)} \subseteq Z \subseteq \overline{O(y)}$ is an intermediate variety.

What is $T_z Z_d$?

Let $\mathcal{G}_{y,d} = \{\mathfrak{g} \in \mathcal{G} | \mathfrak{g}y \in V_{i \geq d}\}$. Then $\pi_d(\mathfrak{g} \cdot y) = T_z Z_d$. How does H_0 act?

The Claims B1 and B2

- There is a suitable extension of y_e into V.
- $dim(\mathcal{H}/\mathcal{H}_{\overline{y_e}})_{(-1)} > 0$ indicates the presence of a $z' \notin O(z)$.

Way Ahead

det_n-the master of all stabilizers

Since all forms f arise out of some det_n, perhaps all stabilizers arise out of a sequence of limits:

$$det_n \xrightarrow{\lambda_1} F_1 \ldots \xrightarrow{\lambda_k} F_k = f$$

Important to analyse how $\mathcal{L}_i = \mathcal{G}_{F_i}$ change.

- Representation Theory and combinatorics
- Stabilizer limits and the data that is associated with it.
- Alignment - the consequences and the hunt.
- Parallels with classical limits
- Deeper orbit-level analysis.

Way Ahead: Codimension 1 forms in $\overline{O(det_n)}$.

det_n-the master of all stabilizers

Since all forms f arise out of some det_n, perhaps all stabilizers arise out of a sequence of limits:

$$det_n \stackrel{\lambda_1}{\to} F_1 \ldots \stackrel{\lambda_k}{\to} F_k = f$$

Important to analyse how $\mathcal{L}_i = \mathcal{G}_{F_i}$ change.

Since the stabilizer of det_n is reductive, the boundary is pure of codimension 1. Suppose these are special forms Q_i. So what is $Q = F_1$ for a form f?

Corollary

Suppose that $W = \overline{O(Q)}$, a component of the boundary, and $Q = \widehat{det_n}^{\lambda}$. Then $\mathcal{L}_1 = \widehat{\mathcal{K}_n} \oplus \ell$ (where $t^{\ell} = \lambda'_1$). Moreover, if there is no alignment, then $\mathcal{L}_1$ is of rank 1.

Let $X = X_3$ be as below and let $det_3(X) \in Sym^3(X)$ be the usual determinant:

$$\lambda_1(t)X_3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & -x_1 - x_5 \end{bmatrix} + t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $X = X_0 \oplus X_1$ where X_0 are trace zero matrices and $X_1 = \mathbb{C}I$, the multiples of the identity.

$$R_1 = \{X \to AXA^{-1}\} \subseteq K_3$$

Note that R_1 commutes with λ_1. Let:

$$\lambda_1(t)det_3 = Q_1 + tQ_1'$$

Then $\mathcal{H}_1 = \mathcal{G}_{Q_1}$ is of dimension 17, $\mathcal{H}_1 = \widehat{\mathcal{K}_3}$, and $Lie(R_1) \subseteq (\mathcal{H}_1)_0$.

$$\widehat{\mathcal{K}_3} = \left[\begin{array}{c|c} * & \mathfrak{u} \\ \hline 0 & \mathfrak{r} \end{array}\right] \quad \overline{k} = \begin{array}{|c|c|c|} \hline -1 & 0 & 1 \\ \hline 8 & 8 & 0 \\ \hline \end{array}$$

8-dimensional alignment.

Let $X = X_3$ be as below and let $det_3(X) \in Sym^3(X)$ be the usual determinant:

$$\lambda_2(t)X_3 = \begin{bmatrix} 0 & -x_3 & -x_7 \\ x_3 & 0 & -x_8 \\ x_7 & x_8 & 0 \end{bmatrix} + t \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_5 & x_6 \\ x_3 & x_6 & x_9 \end{bmatrix}$$

Thus $X = X_a \oplus X_a$ where X_a is the space of anti-symmetric and X_s, symmetric matrices. Let

$$R_2 = \{X \to AXA^T | A \in SL_3\} \subseteq K_3$$

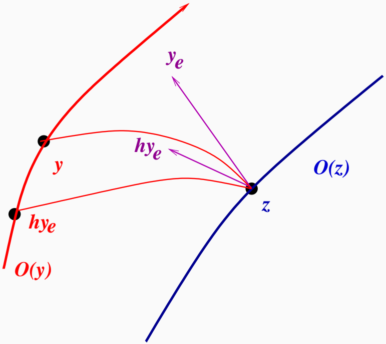
$$\lambda_2(t)det_3 = tQ_2 + t^3 Q_2'$$

Then $\mathcal{H}_2 = \mathcal{G}_{Q_2}$ is of dimension 17, $\mathcal{H}_2 = \widehat{\mathcal{K}_3}$, and $Lie(R_2) \subseteq (\mathcal{H}_2)_0$.

$$\widehat{\mathcal{K}_3} = \left[\begin{array}{c|c} \mathfrak{r}' & \mathfrak{u} \\ \hline 0 & \mathfrak{r} \end{array} \right] \quad \overline{k} =$$

−1	0	1
8	8	0

8-dimensional alignment.



The limit $y \xrightarrow{\lambda} z$ also implies:

- $y \xrightarrow{k\lambda k^{-1}} kz$, for any $k \in K$.
- $hy \xrightarrow{h\lambda h^{-1}} z$, for any $h \in H$.

Let $H_0 \subseteq H$, the subgroup which commutes with λ. Then for an $h \in H_0$, we have:

$$\lambda(t)hy = z + hy_e t^e + \dots hy_D t^D$$

If $\hat{\mathfrak{k}} = \sum_i \mathfrak{k}_i$, then $h\mathfrak{k}h^{-1} = \sum_i h\mathfrak{k}_i h^{-1}$. Thus $h\hat{\mathcal{K}}h^{-1} \subseteq \mathcal{H}_{\overline{hy_e}} \subseteq \mathcal{H}$.

Normalizers

Thus, H_0 acts on the graded objects and the normalizer $N_{H_0}(\hat{\mathcal{K}})$ and $N_{H_0}(\mathcal{H}_{\overline{y_e}})$ have special significance.

Others forms in $\overline{O(det_n)}$

Let $X_m \subset X_n$ as before. Let $A_1, A_2 : X_m \to X_m$ be two linear maps and let B_1, B_2 be the $m \times m$-matrices $B_i = A_i X_m$, i.e., with entries as formal linear combinations of entries of X_m. Let $f_i = det(B_i)$, then $f_i \in \overline{O(det_m)}$. Let G be the $r \times r$-gadget matrix constructed out of B_1 and B_2 such that $det(G) = f_1 + f_2$. Let Y be the $n \times n$-matrix below:

$$\begin{bmatrix} G & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

Then $f = det(Y) = f_1 + f_2 \in Sym^m(X_m)$, is of degree m. The homogenization of f is indeed $f' = x_{nn}^{n-m} f \in Sym^n(X_n)$, and thus $W = \overline{O(f')} \subseteq \overline{O(det_n)}$ and we have the surjection.

$$\mathbb{C}[\overline{O(det_n)}] \twoheadrightarrow \mathbb{C}[W]$$

What are the G-modules in $\mathbb{C}[W]$?