Minimum Fares, Refusals and Empty Travel

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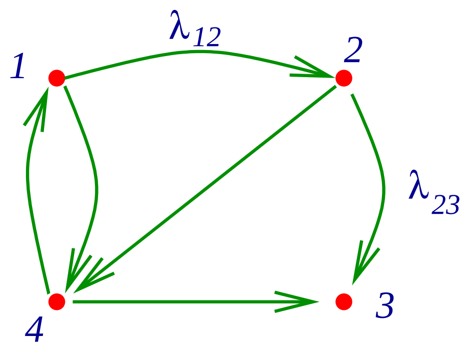
An Attempt

- Minimum Fares: How are minimum fares to be designed? How to ensure that no fares are refused?
- Strategies: What are strategies adopted by trucks/rickshaws, given a fare schedule. When and where do they travel empty.
- Prices: How are prices decided in the transport market? What are cross-subsidies?

The First Model

The basic model consists of N cities/destinations, an $N \times N$ matrix $\Lambda = (\lambda_{ij})$, and another matrix $T = (t_{ij})$.

- λ_{ij}: the arrival rate of customers at i wanting to go to j.
- t_{ij}: the time taken for travelling from i to j.

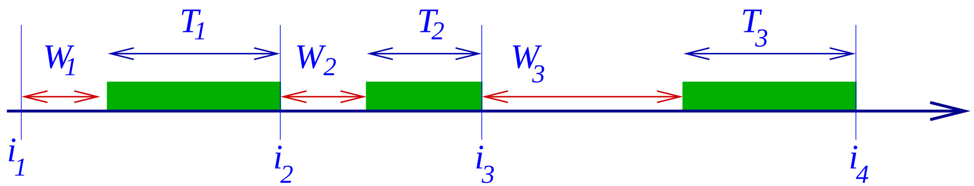


Auto-driver Mr. Pramanik: waits at destination i for the first customer at i. Does not refuse or haggle and takes this customer to her destination j, and then waits at j, and so on...

Question: What are Mr. Pramanik's earnings?

The Earning Rate

So, here is a typical trajectory of Mr. Pramanik:



Obviously, $T_i = t_{i_1,i_2}$, $T_2 = t_{i_2,i_3}$ and so on. We define the earning rate as:

$$ER(\Lambda, T) = \lim_{n\to\infty} \frac{T_1 + T_2 + \ldots + T_n}{W_1 + T_1 + W_2 + T_2 + \ldots + W_n + T_n}$$

Now, we assume that customer arrival is Poisson and that t_{ij} are constants. Then, we construct the markov chain $P = (p_{ij})$, where

$$p_{ij} = \frac{\lambda_{ij}}{\sum_k \lambda_{ik}}$$

Thus, P is the transition matrix for the destinations that Mr. Praminik visits.

An Alternate Expression

Suppose that P is a connected acyclic markov chain and

$$\pi P = \pi$$

is the steady state.
Let $\mu_i = \sum_k \lambda_{ik}$ represent the total outgoing rate at i. Clearly, expected wait at destination i is $w_i = \frac{1}{\mu_i}$.

$$ER(\Lambda, T) = \frac{\sum_{i,j} \pi_i p_{ij} t_{ij}}{\sum_{ij} \pi_i p_{ij} (w_i + t_{ij})}$$

Thus, the earning rate of Mr. Pramanik is completely tractable.

Mr. Chaapter

Mr. Chaapter is more enterprising and considers refusing fares as sanctioned by UN. At destination i, when a customer for j he tosses a coin (with some weight), and accepts the fare only on the coin falling heads.

Question:What is the earning rate of Mr. Chaapter?

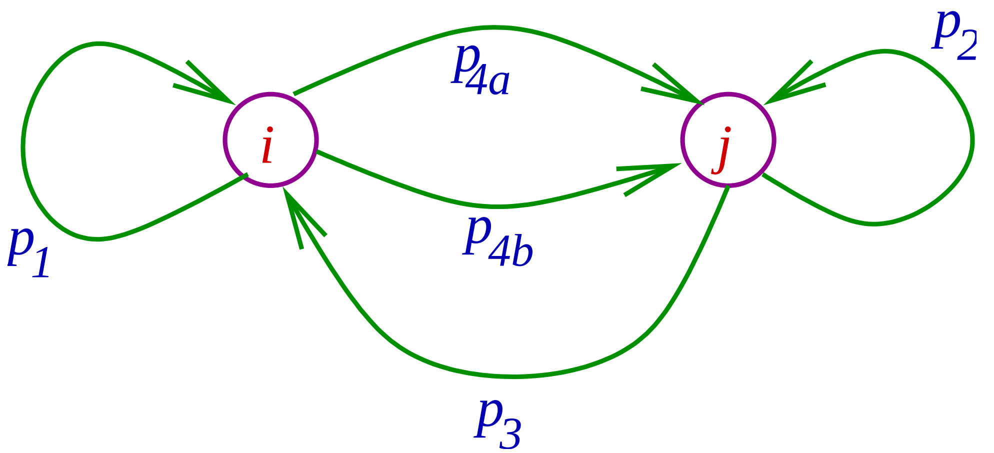
Thus effectively, λ_{ij} is now $\lambda_{ij} - \alpha$ for some α determined by the weight of the coin. Let

$$\Lambda(\alpha) = \Lambda - \alpha E_{ij}$$

What is $ER(\Lambda(\alpha), T)$ and for what values of α is it maximized?

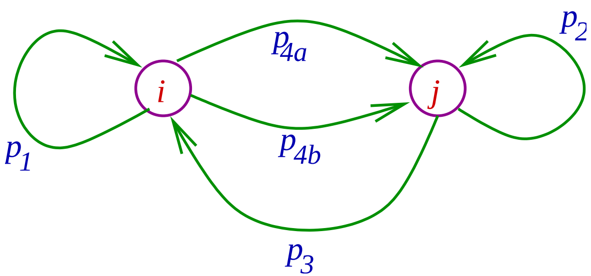
The 2-state chain

For this, we consider the 2-state embedded destination chain:



1	i to i without visiting either of i, j in between	
2	j to j without visiting either of i, j in between	
3	j to i without visiting either of i, j in between	
$4a$	i to j direct	
$4b$	i to j indirect	

Earning Rate thus expressed..



$p_1, p_2, p_3, p_{4a}, p_{4b}$	transition probabilities
$t_1, t_2, t_3, t_{4a}, t_{4b}$	expected travel times
$e_1, e_2, e_3, e_{4a}, e_{4b}$	expected elapse times

Let Π be the staedy state for this embedded chain. In this notation:

$$ER(\Lambda, T) = \frac{\Pi(i)p_1t_1 + \Pi(i)p_{4a}t_{4a} + \Pi(i)p_{4b}t_{4b} + \Pi(j)p_2t_2 + \Pi(j)p_3t_3}{\Pi(i)p_1e_1 + \Pi(i)p_{4a}e_{4a} + \Pi(i)p_{4b}e_{4b} + \Pi(j)p_2e_2 + \Pi(j)p_3e_3}$$

Note that for $\Lambda(\alpha)$, as α changes, the elapse times and the travel times do not change. Thus all that needs to be worked out is the dependence of p_s on α.

. . . as functions of α . . .

Facts:

- Clearly, p_2 and p_3 are unchanged.
- Next, $p_{4a}(\alpha) = \frac{\lambda_{ij}-\alpha}{\mu_i-\alpha}$.
- Finally, $\frac{p_1(\alpha)}{p_{4b}(\alpha)} = \frac{p_1(0)}{p_{4b}(0)}$.

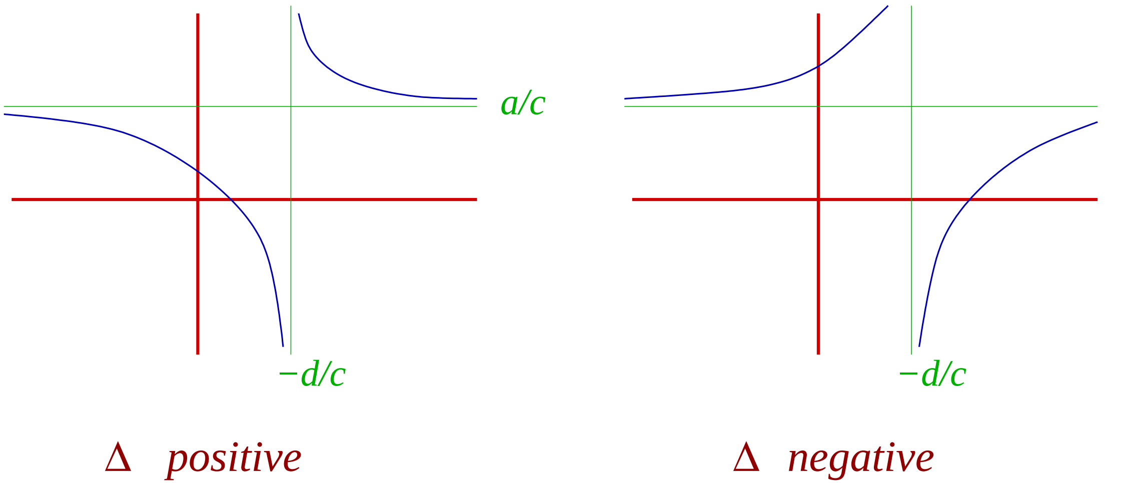
Whence:

$$\begin{aligned} p_1(\alpha) &= \frac{\mu_i p_1(0)}{\mu_i-\alpha} \\ p_{4b}(\alpha) &= \frac{\mu_i p_{4b}(0)}{\mu_i-\alpha} \end{aligned}$$

Whence $\Pi(\alpha)$ may also be calculated. Thus:

$$ER(\Lambda(\alpha), T) = \frac{a\alpha + b}{c\alpha + d}$$

The two possibilities



Thus, we see that, in either case, $\alpha = 0$ or $\alpha = \lambda_{ij}$ is the optimal solution. In other words, refusal is complete, and Mr. Chaapter is Mr. Pramanik on a reduced network.

Empty Returns

Suppose that at destination i, your aunt shows up, wanting to go to j. Clearly, aunts cant be charged!

Are Aunts welcome?

If even one aunt is welcome, then more the merrier.

We do a similar analysis with $\lambda_{ij} \rightarrow \lambda_{ij}+\alpha$, with α non-paying. Again, ER turns out to be rational in α with linear terms. Thus, we see that either $\alpha = 0$ or $\alpha = \infty$.

A Small Conclusion

- Allowing the strategies of refusal and empty returns, the optimal strategy is combinatorial.
- The determination of this optimal strategy seems hard.
- This analysis extends to rupee-earning-rates as well.
- All 4 cases are possible.
- The customer satisfaction rate $S = (s_{ij})$ is proportional to $S_{ij} = \pi_i p_{ij}$.

The Presence of Others

We have so far analysed the earning rates when a single auto-rickshaw is plying in the network. We next consider the case when there are M rickshaws in the network.
We assume:

- There is a queue at each station, where each rickshaw joins upon reaching that station.
- The rickshaw at the head of the queue picks up the arriving customer.

One possibility: Define the state-vector of the system to be a tuple of positive integers $v = (v_1, v_{12}, v_2, \ldots, v_N)$ which encodes the location of each rickshaw. A transition system on this huge state-space may then be defined.

Another possibility

θ_{ij}: the rate of flow of autos between i and j. Let $\theta_i = \sum_k \theta_{ik}$ is the net departure rate at i. As before, P is the transition matrix with $p_{ij} = \frac{\lambda_{ij}}{\mu_i}$. We then have:

$$\sum_r \theta_r p_{ri} = \theta_i$$

Thus the vector $\theta = [\theta_1, \ldots, \theta_N]$ is an eigen-vector of P, whence:

$$\theta = \gamma\pi$$

for some γ.

Assuming (wrongly) the poisson assumption, we have the length of the queue at station i as $\frac{\theta_i}{\mu_i - \theta_i}$. Thus we have the The Truck Inventory Equation

$$\sum_i \frac{\theta_i}{\mu_i - \theta_i} + \sum_{i,j} \theta_i p_{ij} t_{ij} = M$$

The constant γ is easily determined from this equation.

An Example

$$\Lambda = \begin{bmatrix} 0 & 20 \\ 30 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus $\pi = [0.5, 0.5]$. Taking $\gamma = 16$, we get $\theta_1 = \theta_2 = 8$. The truck inventory equation gets us:

$$\frac{8}{20-8} + \frac{8}{30-8} + 8 \cdot 2 + 8 \cdot 3 = 41.030$$

Since there are $8 \cdot 2 + 8 \cdot 3 = 40$ trucks in motion, we have the earning rate as:

$$\frac{40}{41.030} = 0.9748$$

Example continued...

We have the following table:

θ_{21}	θ_{12}	M	ER
8	8	41.030	0.9748
16	16	85.42	0.9398
19	19	115.727	0.8209
19.5	19.5	138.357	0.7047

Thus, as we try and get the last few customers, the waiting costs get higher and higher, and profitability lower and lower.
The operating point will be decided by the capital return rates and the capital mix.

Empty Returns

Next, suppose we allow empty returns δ_{12} from 1 to 2:

θ_{21}	$\theta_{12}-\delta_{12}$	δ_{12}	M	ER
24	18	6	133	0.8120

We explain:

$$\frac{24}{30-24}+\frac{18}{20-18}+24\cdot 2+24\cdot 3=133$$

While the earning autos are: $24\cdot 3+18\cdot 2=108$.
Thus, we see that empty return of 6 is a very good strategy for the same number (roughly 135) of rickshaws.

More Conclusions

- Waiting cost does not scale and alters the profitability.
- Full customer satisfaction is impossible.
- The customer satisfaction θ and the customer arrival μ may differ substantially. This offset is best picked up by empty returns or regulated by Prices.
- Mixed strategies are a MUST

Comment: The real waiting time distribution comes from a closed queueing analysis, see Posner and Bernholtz.

Reluctance...

After this this liberating analysis, why is it that auto-drivers still show some reluctance to go to destinations they dont refuse!

Possible answer: Capital Depreciation

Next, we assume a *per hop depreciation rate* ρ whereby the value of the rupee drops as the number of hops proceed.
Let $s = [s(1), \ldots s(N)]^T$ be a state. Given this, the expected elapse/travel time in the next hop is:

$$
\begin{aligned}
t(s) &= \sum_{i,j} s(i) p_{ij} t_{ij} \\
e(s) &= \sum_{i} s(i) w_i + \sum_{i,j} s(i) p_{ij} t_{ij}
\end{aligned}
$$

Note that both $e(s), t(s)$ are linear in s.

Life-time earnings

Clearly,

- earnings and expenses depreciate geometrically.
- life-time earnings and expenses depend on the initial state s.

Let the auto begin with initial state s^0.
Subsequent states are

$$s^0, s^1, \ldots, s^k = s^0 P^k, \ldots$$

We now compute the total "real" elapse time TRE and total travel time TRT.

Lifetimes...

We have:

$$TRT(\Lambda, T, \rho, s^0) = \sum_{i=0}^{\infty} t(s^i)\rho^i$$
$$TRE(\Lambda, T, \rho, s^0) = \sum_{i=0}^{\infty} e(s^i)\rho^i$$

Next, let $Spec(P) = \{\kappa_1, \ldots, \kappa_N\}$ with eigen-vectors $x_1, \ldots, x_r$ with $\kappa_1 = 1, x_1 = \pi$. Further, we express s^0 as:

$s^0 = \sum_r c_r x_r$. We immediately have:

$$s^i = \sum_r c_r \kappa_r^i x_r$$

Whence:

$$TRT(\rho, s) = \sum_{i=0}^{\infty} t(s^i)\rho^i = \sum_r \sum_i c_r \kappa_r^i \rho^i t(x_r) = \sum_r c_r \frac{t(x_r)}{1 - \rho\kappa_r}$$

The dependence on initial state

Assuming that $\kappa_1 = 1$ and $x_1 = \pi$, we get:

$$TRT(\rho, s) = \frac{t(\pi)}{1-\rho} + c_2 \frac{t(x_2)}{1-\rho\kappa_2} + \ldots + c_N \frac{t(x_N)}{1-\rho\kappa_N}$$

$$TRE(\rho, s) = \frac{e(\pi)}{1-\rho} + c_2 \frac{e(x_2)}{1-\rho\kappa_2} + \ldots + c_N \frac{e(x_N)}{1-\rho\kappa_N}$$

where $s^0 = \sum_r c_r x_r$. Thus, we see that:

- The first term is the steady-state term, while the subsequent terms, all depend on the initial state s^0 via the coefficients c_r.
- As expected, if $\kappa_2 \approx 1$, then this term may dominate earnings.

An Example

$$\Lambda = \begin{bmatrix} 0 & 20 \\ 30 & 0 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

We see that $\kappa_1 = 1$, $x_1 = [0.5, 0.5]$ and $\kappa_2 = -1$ with $x_2 = [0.5, -0.5]$. Further:

x	$t(x)$	$e(x)$
x_1	2.5	2.583
x_2	-0.5	-0.492

Then, with $\rho = 0.9$, we have for the pure states:

s	TRT	TRE	ER
$[1, 0]$	24.737	25.571	0.9674
$[0, 1]$	25.263	26.089	0.9683

Thus, the driver would doubly prefer destination 2. More earnings over a longer time!

Demand Satisfaction

Coming back: demand satisfaction is the matrix

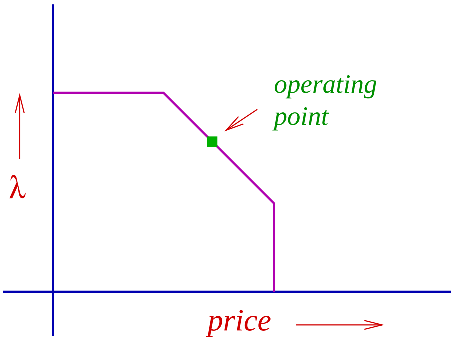
$$S = (s_{ij}) = (\theta_i p_{ij})$$

This may be quite different from Λ, the customer demand. There are two variables which handle this difference.

- Empty returns: This is a variable which may be used to offset waiting costs of demand-satisfaction by transit costs.
- Prices: these may be used to control demand itself. Thus, if demand satisfaction at i is poor, then prices out of i may increase and prices into i may decrease.

We conducted a few numerical experiments when the arrival rate Λ was elastic to prices. We observed the above scenarios and expected behaviours.

The Numerical Model



Variables		Parameters	
θ_{ij}	fared travel	λ_{ij}	functions of y_{ij}
η_{ij}	empty travel	c_{ij}	cost incurred in travel
y_{ij}	price proposed	z_i	cost of waiting

The formulation

$$
\begin{aligned}
0 &\le \theta_{ij} \le \lambda_{ij} \\
0 &\le \eta_{ij} \\
l_{ij} &\le y_{ij} \le u_{ij}
\end{aligned}
$$

Flow conservation:

$$\sum_r (\theta_{ri} + \eta_{ri}) = \sum_s (\theta_{is} + \eta_{is})$$

Truck Inventory:

$$\sum_i \frac{\theta_i}{\mu_i - \theta_i} + \sum_{i,j} (\theta_{ij} + \eta_{ij}) t_{ij} = M$$

Objective:

$$\sum_i \frac{z_i \theta_i}{\mu_i(\mu_i - \theta_i)} - \sum_{i,j} (\theta_{ij} + \eta_{ij}) t_{ij} c_{ij} + \sum_{i,j} \theta_{i,j} p_{ij}$$

Conclusions-Final

We thus have a computational model of the system:

- Network wide model which includes fleet-size.
- Models waiting costs and cost of meeting demand.
- Allows for reasonable strategies of refusal and empty travel.
- Concise: number of varibles of the order of network size.
- Seems to be of predictive value.