Winding Number

Lecture Slides By
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Introduction

- Orientation and Gauss Map: Consider a curve

  $C : [a, b] \to \Re^3$
  where
  $C(a) = C(b)$

  an orientation of C is a map $\overrightarrow{o} : C \to S^2$ such that:

  – $\overrightarrow{o}$ is continuous and smooth
  – for all non-singular $p \in C, \overrightarrow{o}(p) \in TC_p$.

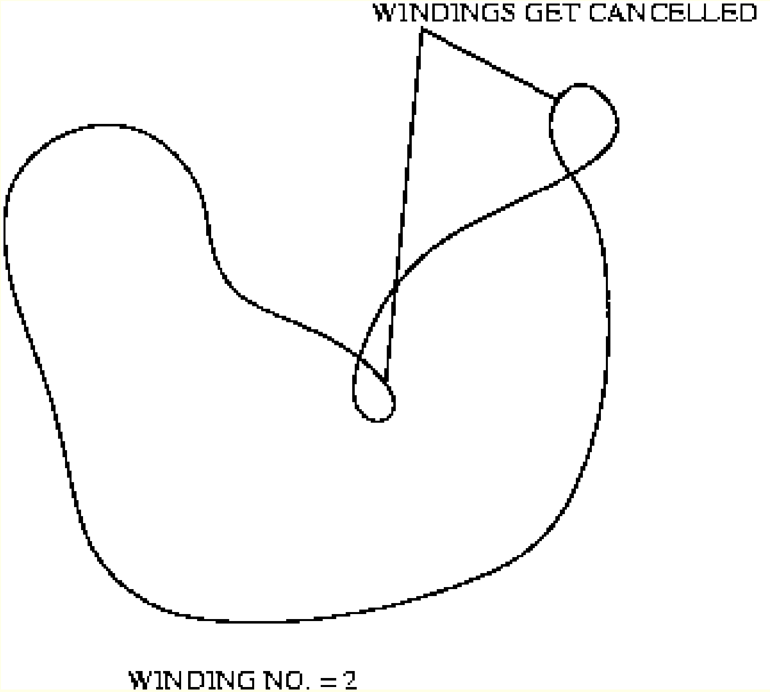
  Thus an orientation $\overrightarrow{o}(t)$ is a continuously varying unit velocity vector. As we traverse the curve from a to b, $\overrightarrow{o}(t)$ describes a curve that remains entirely in the unit circle. The mapping $t \to \overrightarrow{o}(t)$ is called the Gauss map for the curve $t \to \overrightarrow{c}(t)$.

Winding Number

- If we imagine that $\overrightarrow{o}(t)$ carries a string with itself that it lets out as it moves. When it returns to the starting point, the string may have have wrapped once around the circuit or two or three times or it might not be wrapped at all.
- In each case an integer can be determined, the number of times the string wrapped around the unit circle. This integer is an invariant of the curve and yields the winding number.
- If the curve does not cross itself, the winding number is $+/-2$.
- If it crosses itself once, it can be 0 or $+/-2$.
- An intuitive way of checking for winding number would be to determine the number of times a curve loops. If a curve can be unraveled into a single loop , its winding number will be 2.

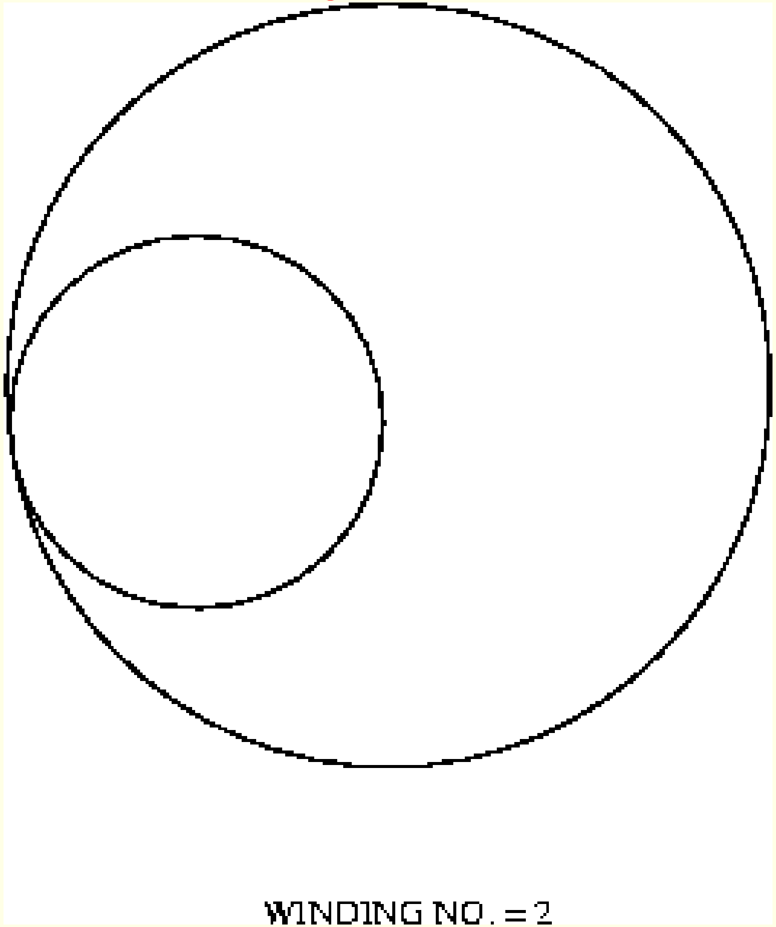
Example

Winding Number = 1



Another Example

Winding Number = 2



Curvature

- Consider a fixed orientation $\overrightarrow{o}$ for a curve $C \in \Re^3$. Let $\overrightarrow{o}(p) = \overrightarrow{v}$ be the unit tangent vector at p.
- The map $\overrightarrow{o}$ translates to a linear map $\overrightarrow{o}^*_p : TC_p \to TS^2_v$.
- Let t_0 be such that $\overrightarrow{c}(t_0) = p$ then for infinitesimal motion ϵt $\overrightarrow{c}(t_0 + \epsilon\text{t}) = \overrightarrow{c}(t_0) + \epsilon\text{t}\overrightarrow{c}_t(t_0)$.
- This small motion produces the change $\overrightarrow{o}(t_0{+}\epsilon\text{t}) = \overrightarrow{o}(t_0) + \epsilon\text{t}\overrightarrow{o}_t(t_0)$.
- Thus the element $\overrightarrow{c}_t(t_0) \in TC_p$ produces the change $\overrightarrow{o}_t(t_0) \in TS^2_v$.
- The vector $\frac{\overrightarrow{o}_t(t_0)}{\|\overrightarrow{c}_t(t_0)\|}$ is known as the **curvature vector**, its magnitude curvature, and is denoted by κ.

Curvature and winding

- The idea behind the *Gauss map* which maps a curve C to a unit circle S is that it "wraps" the curve around the circle an integer number of times.
- The arc on the unit circle through which $\overrightarrow{o}(t)$ turns can be obtained by integrating the curvature $\kappa(t)$.
- $\int_C \kappa =$ **the total winding of** $\overrightarrow{o}(t)$
$=$ a multiple of 2π for a general closed curve.
- This idea will be illustrated more clearly in the proof following the example..

Example

- To calculate the winding number for a circle C:($r\cos t$, $r\sin t$) where t $\in [0, 2\pi]$ and r the radius of the circle.
- Orientation $\overrightarrow{o}(t) = (-\sin t, \cos t)$
- Therefore curvature vector $\overrightarrow{\kappa}(t) = (-\cos t, -\sin t)$.
- Therefore integrating the curvature over $[0, 2\pi]$ we get $\int_0^{2\pi}(-\cos t, -\sin t).(\cos t, \sin t)dt$ which is an integral multiple of 2π.

Proof

- Suppose the velocity frame of the curve $\overrightarrow{c}(t)$ is described by orthogonal unit vectors $\overrightarrow{v}(t)$ and $\overrightarrow{\gamma}(t)$.

- $\overrightarrow{v}'(t) = \kappa(t)\overrightarrow{\gamma}(t)$
  $\gamma'(t) = \alpha\overrightarrow{v}(t) + \beta\overrightarrow{\gamma}(t)$
  $\alpha = \langle\overrightarrow{\gamma}'(t), \overrightarrow{v}(t)\rangle = -\langle\overrightarrow{v}'(t), \overrightarrow{\gamma}(t)\rangle = -\kappa(t)$

- Now let $\overrightarrow{v}(t) = (\cos\theta(t), \sin\theta(t))$ and $\overrightarrow{\gamma}(t) = (-\sin\theta(t), \cos\theta(t))$ where θ can be intuitively defined as the turning angle.

- If $a =: x_0 <= x_1 <= ..... <= x_{n-1} <= x_n := b$ then $\theta_i(x)$ is the angle in [0,2π], measured counterclockwise, between $\overrightarrow{v}(x_{i-1})$ and $\overrightarrow{v}(x_i)$.

Winding Number

- $-\kappa(t) = -(\sin^2\theta(t) + \cos^2\theta(t))\frac{d\theta}{dt} = \frac{d\theta(t)}{dt}$
- The *total signed curvature* of a curve C:[a,b] is given by $\int_C \kappa(t)dt = \int_a^b \theta'(t)dt = \theta(b) - \theta(a)$.
- For a closed curve therefore the total winding is a multiple of 2π, since $\overrightarrow{v}(a) = \overrightarrow{v}(b)$.
- Thus, *Total Winding* $= m * 2\pi$ where m denotes the winding number.