Euclidean Geometry

Let \( v = (x, y, z) \) be a typical element of \( \mathbb{R}^3 \). We define \( |v| \) as \( \sqrt{x^2 + y^2 + z^2} \). Given two vector \( v = (x, y, z) \) and \( w = (x', y', z') \), the inner product is defined as \( (v, w) = xx' + yy' + zz' \). Vector \( v \) is called **orthogonal** to \( w \) if \( (v, w) = 0 \). Vector \( v \) is called a **unit vector** if \( |v| = 1 \).

1. Show that \( (v, w) \leq |v||w| \).

2. Recall that the cross product \( v \times w \) is defined as the vector

\[
\begin{vmatrix}
  i & j & k \\
  x & y & z \\
  x' & y' & z'
\end{vmatrix}
\]

Prove that \( v \times w \) is orthogonal to both \( v \) and \( w \).

3. A Line in \( \mathbb{R}^3 \) may be represented as a tuple \( (v, w) \) with both \( v, w \in \mathbb{R}^3 \). This denotes the line \( v + tw \) where \( t \in \mathbb{R} \). (i) Given \( L_1 = (v_1, w_1) \) and \( L_2 = (v_2, w_2) \), determine if these lines are identical or parallel. (ii) Outline a test to determine if a point \( p \) lies on \( L \).

4. Show that if two lines are neither parallel nor intersecting, then there are two parallel planes each containing one of the lines. Is such a pair unique?

5. A plane is given by the equation \( ax + by + cz + d = 0 \). Let \( p \) be a point in \( \mathbb{R}^3 \). Give an algorithm to find the closest point \( q \) on the plane to \( p \).

6. Determine if a line \( L = (v, w) \) lies on a plane of the above form.

7. Let \( P \) and \( Q \) be two planes given by the equations \( ax + by + cz + d = 0 \) and \( a'x + b'y + c'z + d' = 0 \). Compute the intersection of \( P \) and \( Q \).

8. Consider the vector \( w = [1, 2, 3] \). Compute a basis for the space \( \{v| (v, w) = 0 \} \), of all vectors orthogonal to \( w \).

9. Let \( P = \{p_1, \ldots, p_r\} \) be a collection of points. Let \( Q = \{q_1, \ldots, q_s\} \) be another collection so that each \( q_i \) is a convex combination of elements of \( P \). Now let \( r \) be a convex combination of elements of \( Q \). Show that \( r \) is also a convex combination on \( P \).

10. Let \( p_1, p_2, p_3 \) be distinct points. Let \( T \) be the collection of all convex combinations of these points. Show that \( T \) is a triangle with vertices \( p_1, p_2, p_3 \). Show that every point in \( T \) has a unique expression as a convex combination of the points \( p_1, p_2 \) and \( p_3 \).

11. Repeat the above problem with 4 points. **Caution:** Some things are different!

12. A \( 3 \times 3 \) matrix \( X \) is called a **rotation matrix** if \( XX^T = I \), the identity matrix. We say that \( v_1, v_2, v_3 \) form an **orthonormal** system if all are unit vectors and orthogonal to each other.

Show that if \( X \) is a rotation matrix, then its rows form an orthonormal system. Show the converse. Show that if \( X, Y \) are rotation matrices, then so are \( XYX^T = XYX^{-1} \) and \( X^TYX = X^{-1}YX \).
13. Show that the following matrix $Z(\theta)$ is a rotation matrix.

$$
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

Recall that a matrix defines a linear transformation on $\mathbb{R}^3$ treated as row-vectors. Show that $[0, 0, 1]Z(\theta) = [0, 0, 1]$, and thus the $Z$-axis is left invariant by $Z(\theta)$.

14. Construct matrices $X(\theta)$ and $Y(\theta)$ with the appropriate properties. Let $R$ be a rotation matrix with rows $v_1, v_2, v_3$. Show that $v_3 R^T Z(\theta)R = v_3$. Describe geometrically, this matrix.

15. Let $v$ be a unit vector. Construct a rotation matrix $R$ such that $vR = [1, 0, 0]$.

Polynomials

1. Show that $T_a = \{1, (t-a)^1, \ldots, (t-a)^n\}$ is a basis for $P^n[t]$.

2. Let $p$ be a polynomial. Suppose that $p(a) = 0$, then argue that $(t-a)$ divides $p$.

3. Evaluate $\int B^n_i(t)dt$ and $\frac{dB^n_i(t)}{dt}$. What is the maximum value of $B^n_i(t)$ on $[0, 1]$?

4. Prove the degree elevation and subdivision formulae given in the class.

5. Construct a cubic polynomial $p$ such that $p(0) = p'(0) = p(1) = 0$ and $p'(1) = 1$. This polynomial is one of the Hermite polynomials.

6. Argue for the linear independence of the Bernstein and the Lagrange Basis. Hint for the lagrange basis: show the invertibility of the van der Waerden matrix:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
a & b & c & d \\
a^2 & b^2 & c^2 & d^2 \\
a^3 & b^3 & c^3 & d^3
\end{bmatrix}
$$

7. Construct $B^3(f)$, the 3-rd degree bernstein approximation to the function $f(t) = t^2$. How close is $B^3(f)$ to $f$?

8. Do the same for the Lagrange interpolator $L^3(f)$ with $f = t^2$.

Curves and Surfaces

1. Let $f : [-1, 1]$ be defined piece-wise as follows:

$$
f(t) = \begin{cases} 
t & t \in [-1, 0] \\
\sin t & t \in [0, 1]
\end{cases}
$$

Compute the order of continuity of $f$ at 0.

2. Consider the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$
t \rightarrow \left( \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right)
$$

Show that image $f(t)$ of any point $t$ lies on the unit circle. Compute $f(0), f(\pm 1), f(\pm 2)$. Is there any point on the unit circle which is not an image of $f$?
3. Let $S$ be a given by an equation $f(X,Y,Z) = 0$. The gradient of the function $f$ is given by the sequence of functions

$$\nabla(f) = \left( \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z} \right)$$

Compute $\nabla(f)$ for the function $X^2 + Y^2 + Z^2 - 1$ at the point $[1, 0, 0]$.

4. Let $\epsilon$ be so small a quantity such that $\epsilon^2 = 0$. Let a surface $S$ be given by $f(X,Y,Z) = 0$ and let $p = (x_0, y_0, z_0)$ be a point on $S$. Let $q = (x, y, z)$ be another vector. Let us consider all vectors $q$ so that $f(p + \epsilon q) = 0$. Such $q$ will be called tangent vectors at $p$.

Compute the space of tangent vectors for the point $[1, 0, 0]$ on the unit sphere. As an example, we see that $[0, 0, 1]$ is a tangent since we have $p + \epsilon q = [1, 0, \epsilon]$. Substituting this in $X^2 + Y^2 + Z^2 - 1 = 0$, we see that $p + \epsilon q$ does indeed satisfy the equation.

Compute the tangent plane on a generic point on the cone $X^2 + Y^2 - Z^2 = 0$. Is there any relationship between the tangent space at a point and the gradient there?

5. Compute the implicit form for the torus (as a polynomial).

6. Let $C$ be a Bezier Curve with control polygon $P = [p_0, \ldots, p_n]$. Show that an affine transformation (i.e., a translation and/or a homogenous linear transformation) of the curve is obtained by applying the same transformation to the control points.

7. A soap tablet has been specified by its cross-sections

Construct bezier surface patches to match these specifications.

8. Let $P$ be the set of control points for a cubic bezier curve $C$ as shown below:

<table>
<thead>
<tr>
<th>$p_0$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, 0, 0]$</td>
<td>$[0, 1, 0]$</td>
<td>$[1, 1, 0]$</td>
<td>$[2, 0, 0]$</td>
</tr>
</tbody>
</table>

(i) Evaluate $C(0.5)$ and subdivide $C$ to 0.5.
(ii) Elevate the degree of $C$ to 4.

9. Consider a quadratic bezier surface given by the following control points:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$[2, 0, 0]$</td>
<td>$[2, 1, 0]$</td>
<td>$[2, 1, 3]$</td>
</tr>
<tr>
<td>$[1, 0, 0]$</td>
<td>$[1, 1, 0]$</td>
<td>$[1, 1, 2]$</td>
</tr>
<tr>
<td>$[0, 0, 0]$</td>
<td>$[0, 1, 0]$</td>
<td>$[0, 1, 1]$</td>
</tr>
</tbody>
</table>
Compute \( S(0.5, 0.5) \). Also elevated the \( u \) degree to 3.

10. We are given the knot vector \([0, 0, 0, 2, 3, 3]\), and control points:

\[
\begin{array}{cccccc}
\hline
p_1 & p_2 & p_3 & p_4 & p_5 \\
[0, 0, 0] & [0, 1, 0] & [1, 1, 0] & [2, 0, 0] & [3, 0, 0] \\
\hline
\end{array}
\]

Evaluate this B-spline at \( t = 1 \).

Constructions and Operations

1. Let \( f(u, v) = u^2 + u + 2v \) and \( g(u, v) = v^2 + 2u + v \). Starting from the initial guess of \((1, 1)\), use the Newton-Raphson technique to compute the next two iterations.

2. Formulate a procedure for creating surfaces of revolution.

3. Consider the situation of a drafted extrude where the profile has sharp corners. Describe the geometry/topology near these sharp corners.

4. Argue why the surface line on the blend surface discussed in the class is indeed so.

5. Given two points on a unit sphere, derive the parametrization of the great circle passing through it.

6. Let \( S \) be the unit cube and let \( e_1, e_2, e_3 \) be the edges incident at a vertex. Suppose \( e_1, e_2 \) are blended first with radius \( r \) and \( e_3 \) subsequently with radius \( R \). Describe the geometry of all the surfaces created. Cover the cases when \( r < R \) and \( r > R \) separately. Describe what happens when this sequence is reversed.