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Surfaces:Tensor Products

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Polynomials in 2 variables

Let $P^{m,n}[u,v]$ denote the vector space of all polynomials of degree atmost m in u and n in v. Thus, for example,

$$3u^2v - v^3 \in P^{2,3}[u,v] \subset P^{3,3}[u,v]$$

The dimension of $P^{m,n}[u,v]$ is obviously $(m+1)(n+1)$ and the Taylor basis for it is the set:

$$\{u^iv^j | 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

Just as polynomials in one variable served us to parametrize curves, these will serve us to parametrize surfaces.

Tensor-Product Bases

Actually, if $B = \{b_0(u), \ldots, b_m(u)\}$ is a basis for $P^m[u]$ and $C = \{c_0(v), \ldots, c_n(v)\}$ is a basis for $P^n[v]$ then:

$$B \otimes C = \{b_i(u)c_j(v) | 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

is a basis for $P^{m,n}[u, v]$.
Question :Show that elements of $B \otimes C$ are linearly independent.
Suppose that (as polynomials):

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij} b_i(u) c_j(v) = 0$$

Whence, for every u_0, we construct the polynomial:

$$p(u_0, v) = \sum_{j=0}^{n} (\sum_{i=0}^{m} \alpha_{ij} b_i(u_0)) c_j(v)$$

We see that $p(u_0, v) = 0$ for all v, whence *every* coefficient of $p(u_0, v)$ must be zero. In other words, for all j and u_0,

$$\sum_{i=0}^{m} \alpha_{ij} b_i(u_0) = 0$$

Since, b_i's are linearly independent, we are forced to conclude that $\alpha_{ij} = 0$ for all i and j.

□

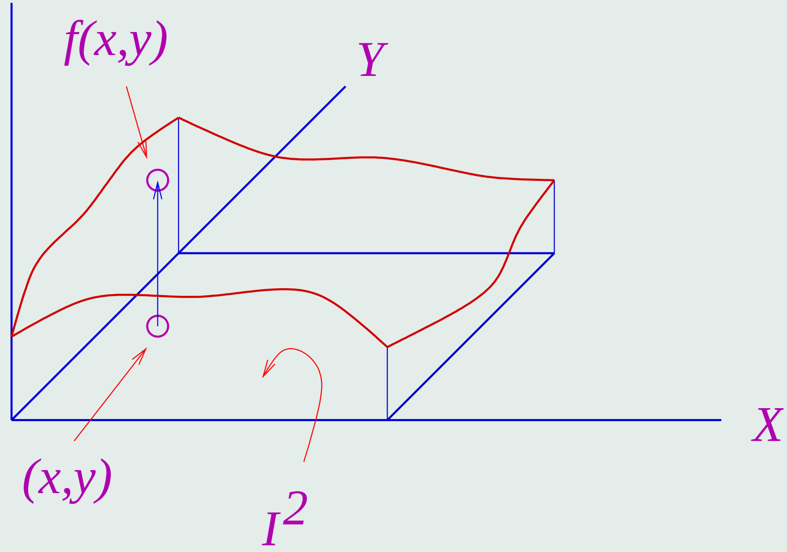
In particular we have the: Bernstein Basis:

$$\{\binom{m}{i} u^i (1-u)^{m-i} \binom{n}{j} v^j (1-v)^{n-j} | 0 \le i \le m,\ 0 \le j \le n\}$$

We denote the typical basis element by $B_i^m(u) B_j^n(v)$.

Functions and the Approximation Problem

I with denote the interval $[0, 1]$ and I^2 the unit square $[0, 1] \times [0, 1]$. Let $f : I^2 \longrightarrow \mathbb{R}$ be a function on the unit square.



Is there a polynomial approximation to f?

The Bernstein-Weierstrass Approximation Theorem

Fix m and n, and form the data

$$S = \{f_{ij} = f(\frac{i}{m}, \frac{j}{n}) | 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

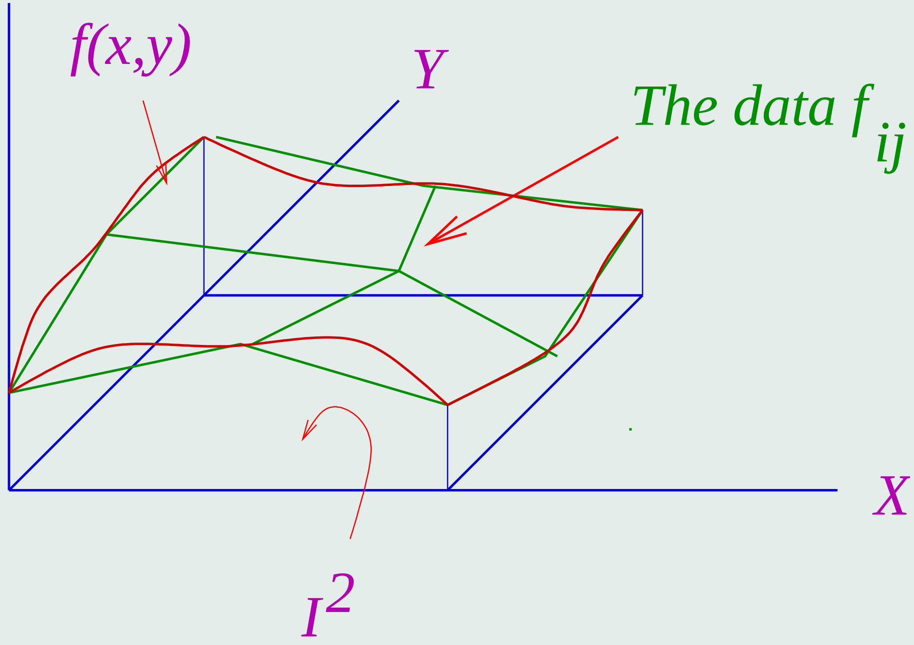
We define the Bernstein Approximation

$$B^{m,n}(f)(u, v) = \sum_i \sum_j f_{ij} B_i^m(u) B_j^n(v)$$

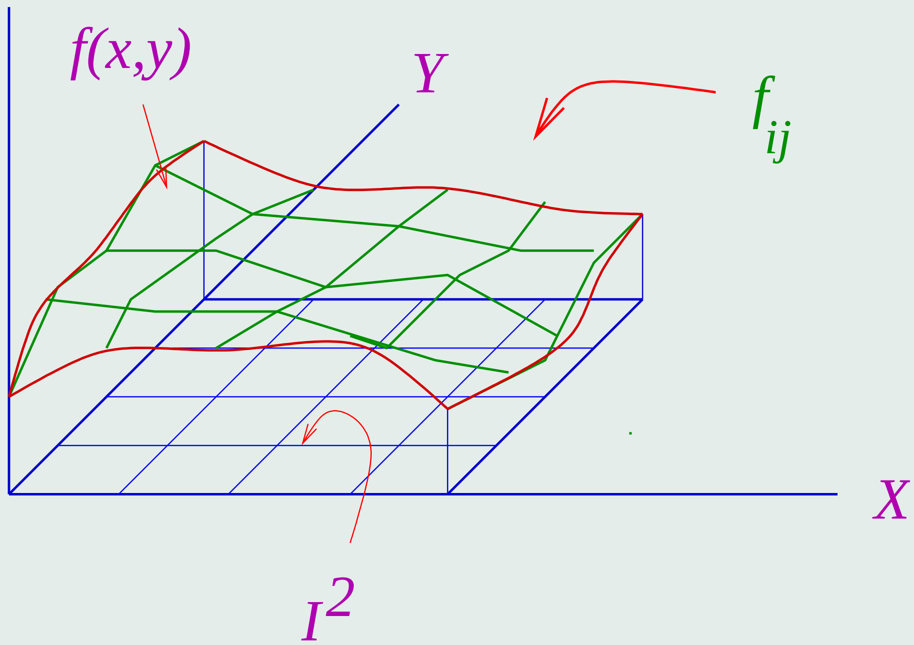
Theorem: Let f be a function on I^2, and let $\epsilon > 0$. Then there are m, n such that $|f(u, v) - B^{m,n}(f)(u, v)| < \epsilon$ for all $(u, v) \in I^2$.

Thus the 1-d situation has a complete 2-d analogue.

The Picture

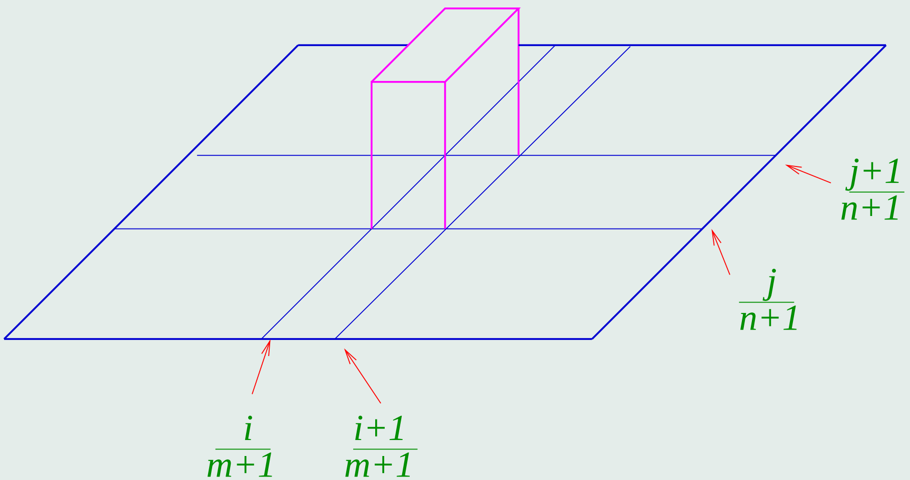


The Finer Picture



The Unit Step

As before it si convenient to associate $B_i^m(u)B_j^n(v)$ with the 2-dimensional unit step function below. The 'greville abscissa' is obviously $(\frac{i}{m}, \frac{j}{n})$ which occurs within the support of the step.



As expected $\int_0^1 \int_0^1 B_i^m(u)B_j^n(v)dudv = \frac{1}{(m+1)(n+1)}$.

The Control Polygon

We will now discard the function f.

Let S be an $m \times n$ matrix (in C++ notation, i.e., [0...m][0...n]) with entries in $\mathbb{R}$ (or $\mathbb{R}^3$).

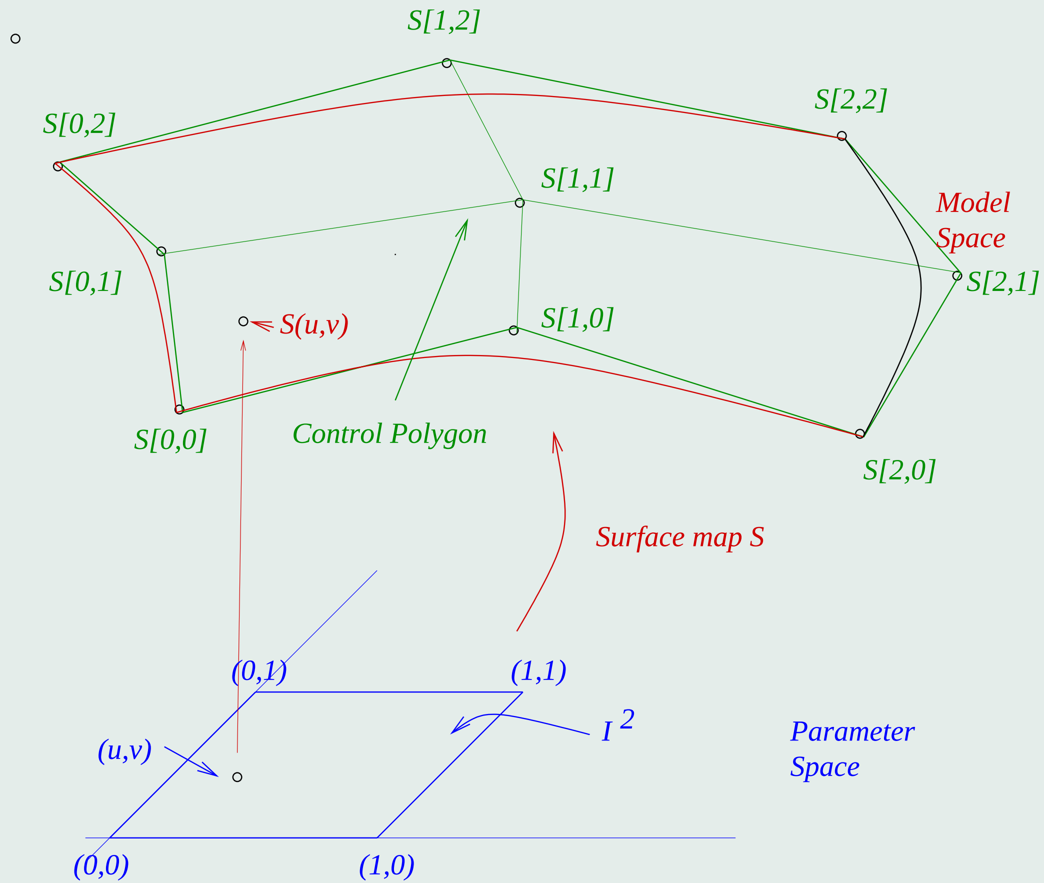
S is called the Control Polygon.

We define $S(u, v)$ as:

$$S(u, v) = \sum_i \sum_j S[i, j] B_i^m(u) B_j^n(v)$$

S will be called the tensor-product surface for the given control polygon.

An example


S[1,2]
S[0,2]
S[2,2]
S[1,1]
Model
Space
S[0,1]
S[2,1]
S(u,v)
S[1,0]
S[0,0]
Control Polygon
S[2,0]
Surface map S
(0,1)
(1,1)
I^2
(u,v)
Parameter
Space
(0,0)
(1,0)

Some Observations

$$\boxed{S(u,v) = \sum_i \sum_j S[i,j] B_i^m(u) B_j^n(v)}$$

Lets evaluate $S(0,0)$. Since $B_i^m(0) = 0$ unless $i = 0$ and $B_j^m(0) = 0$ unless $j = 0$, we have $S(0,0) = S[0,0]$. Similarly, we have the other ‘corner points’. Thus:

$$\boxed{\begin{aligned} S(0,0) &= S[0,0] \\ S(1,0) &= S[m,0] \\ S(0,1) &= S[0,n] \\ S(1,1) &= S[m,n] \end{aligned}}$$

Boundary Curves

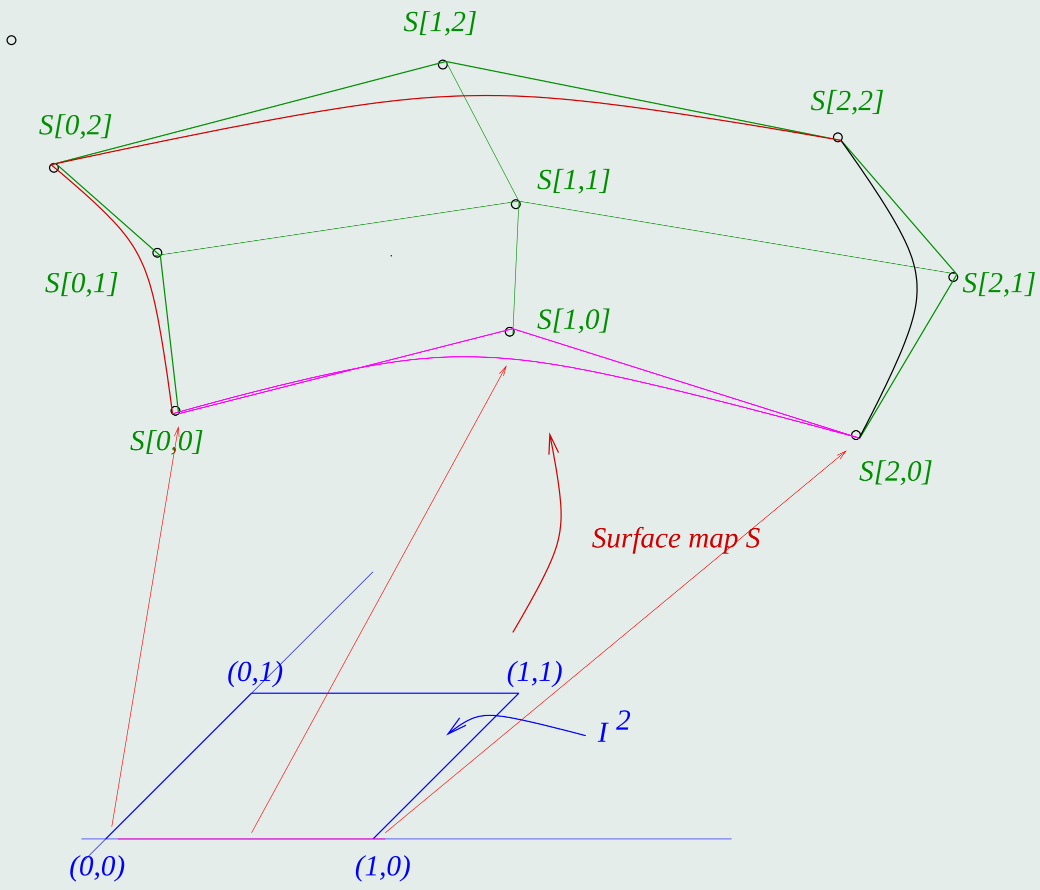
Next, lets look at $S(u, 0)$, which is the image of a boundary line of I^2. Again, since on this curve $v = 0$, we have $B_j^n(0) = 0$ for $j \neq 0$. Thus the sum reduces to:

$$S(u, 0) = \sum_{i=0}^{m} S[i, 0] B_i^m(u)$$

This is clearly the bezier curve corresponding to the first column of S as its control points.
In general, we have:

$$\boxed{\begin{array}{lcl} S(u, 0) & = & \sum_{i=0}^{m} S[i, 0] B_i^m(u) \\ S(u, 1) & = & \sum_{i=0}^{m} S[i, n] B_i^m(u) \\ S(0, v) & = & \sum_{j=0}^{n} S[0, j] B_j^n(v) \\ S(1, v) & = & \sum_{j=0}^{n} S[m, j] B_j^n(v) \end{array}}$$

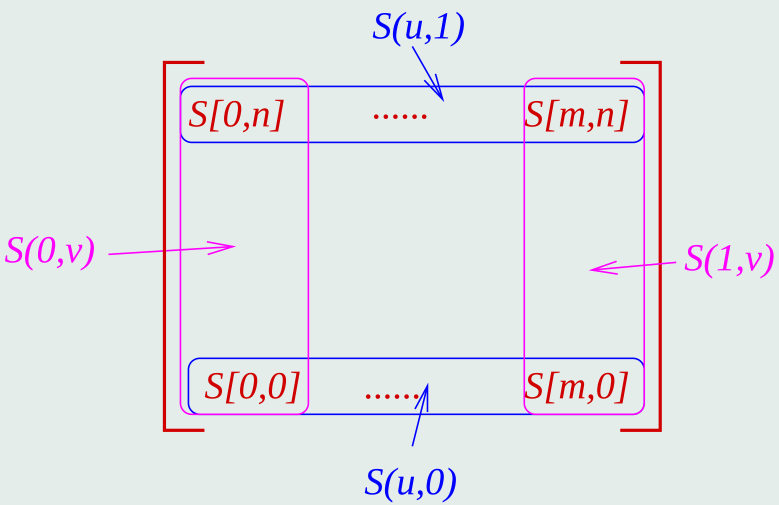
Pictorially



And Schematically

In terms of the control matrix, perhaps it is usefule to use the *french notation* and number rows and columns from the bottom left corner. Then, we have:

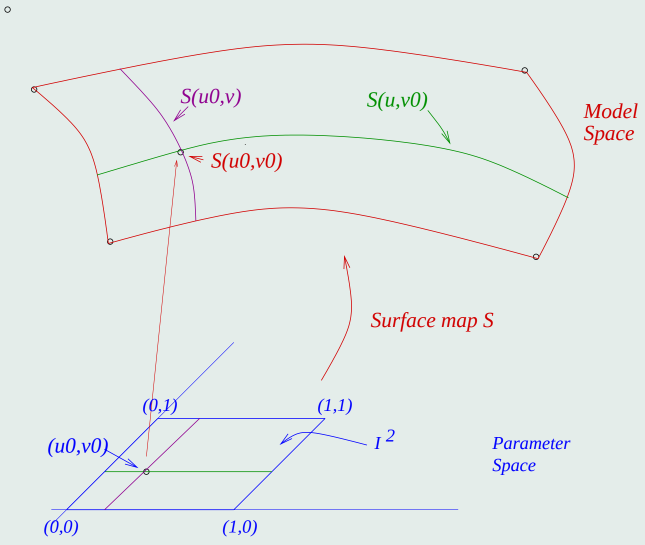
The Control Matrix S



Iso-parametric Lines

But what about general $S(u_0, v)$ for a fixed u_0 and $v \in [0, 1]$? Or $S[u, v_0]$ for a fixed v_0 but u ranging over $[0, 1]$?

These curves (in the model space) are called iso-parametric lines. Thus $S[u_0, v]$ is the iso-parametric line for $u = u_0$.



Iso-parametric Lines contd.

Lets evaluate $S(u_0, v)$. Re-arranging the sum $S(u, v)$, we see that:

$$S(u_0, v) = \sum_{j=0}^{n}[\sum_{i=0}^{m} S[i, j]B_i^m(u_0)]B_j^n(v)$$

We call $\sum_{i=0}^{m} S[i, j]B_i^m(u_0)$ as $S[u_0, j]$ and observe that $S(u_0, v)$ is a bezier curve with control points $[S[u_0, 0], S[u_0, 1], \ldots S[u_0, n]]$.

Also, note that *each* of these control points $S[u_0, j]$ is itself moving on a bezier curve parametrized by u.

Perhaps, the matrix notation is more convenient to observe this. We see that:

$$S(u,v) = [B_n^n(v), \ldots, B_0^n(v)] \begin{bmatrix} S[0,n] & \ldots & S[m,n] \\ \vdots & & \vdots \\ S[0,0] & \ldots & S[m,0] \end{bmatrix} \begin{bmatrix} B_0^m(u) \\ \vdots \\ B_m^m(u) \end{bmatrix}$$

This may be consicely written as $S(u,v) = B(v)SB(u)^T$. Consequently, forming the product as $S(u,v) = B(v)(SB(u)^T)$, we see that:

$$S(u_0,v) == [B_n^n(v), \ldots, B_0^n(v)] \begin{bmatrix} S[u_0,n] \\ \vdots \\ S[u_0,0] \end{bmatrix}$$

Also note that $\sum_i \sum_j B_i^m(u)B_j^n(v) = 1$ and thus $S(u,v)$ is a convex combination of the entries of S.

End Tangents and Normals

Given a map $S : I^2 \to \mathbb{R}^3$ as we have already determined the boundary $S(0, v), S(u, 0)$, and so on. Other important data is the first-order data, viz., the tangents.
For convenience, let us consider the boundary point $S(u_0, 0)$. At any boundary point, we have two tangents to compute.

$$\begin{aligned} S_u(u_0, 0) &= \lim_{u \to u_0} \frac{S(u,0) - S(u_0,0)}{u - u_0} \\ S_v(u_0, 0) &= \lim_{v \to u_0} \frac{S(u_0,v) - S(u_0,0)}{v} \end{aligned}$$

These two tangents are shown in the next picture.

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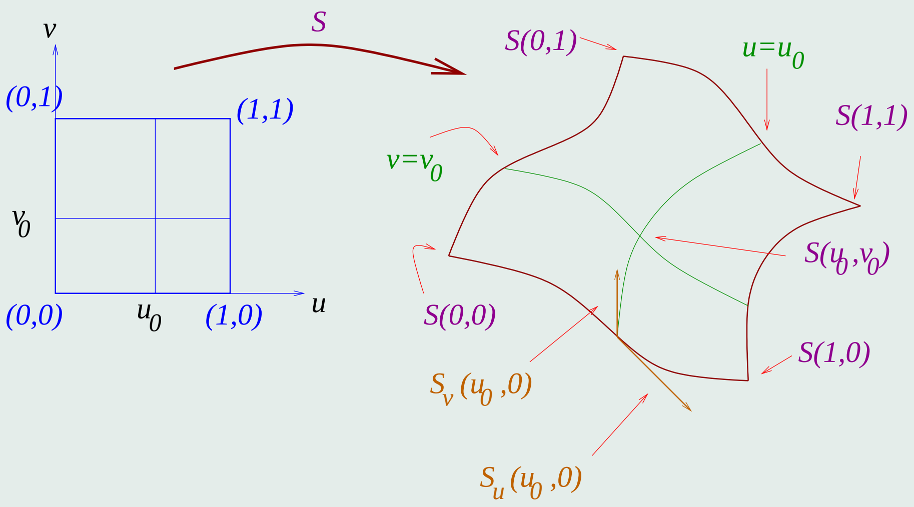
An Example



The quantity $S_u(u_0, 0)$ is easily computed as the derivative of the boundary $S(u, 0) = \sum_{i=0}^{m} S[i, 0]B_i^m(u)$. We may thus use the curve-tangent law explained earlier to get:

$$S_u(u_0, 0) = m[\sum_{i=0}^{m-1}(S[i+1, 0] - S[i, 0])B_i^{m-1}(u)]$$

$$S_v(u_0, v)$$

This quantity is a bit more delicate, since it is the tangent to the iso-parametric curve $S(u_0, v)$ at $v = 0$.
We have seen that:

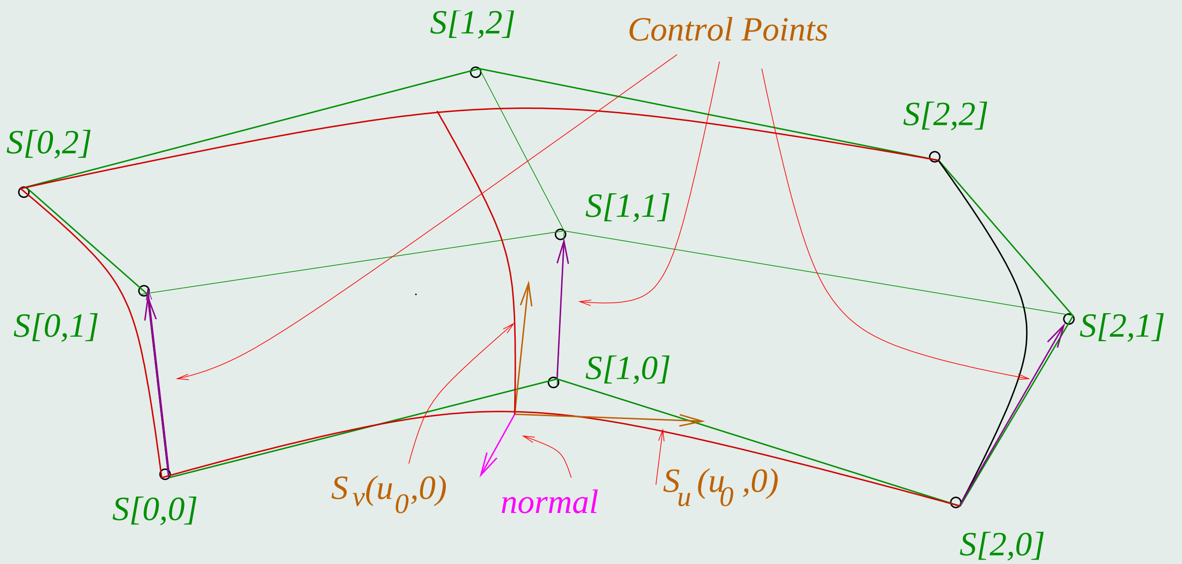
$$S(u_0, v) = \sum_{j=0}^{n} S[u_0, j] B_j^n(v)$$

where $S[u_0, j] = \sum_{i=0}^{n} S[i, j] B_i^m(u_0)$.
Thus $S_v(u_0, 0)$, the end-tangent to this curve, is $m(S[u_0, 1] - S[u_0, 0])$.
Back-substituting, we get:

$$\begin{aligned} S_v(u_0, 0) &= m[\sum_{i=0}^{n} S[i, 1] B_i^m(u_0) - \sum_{i=0}^{n} S[i, 0] B_i^m(u_0)] \\ &= m[\sum_{i=0}^{n} (S[i, 1] - S[i, 0]) B_i^m(u_0)] \end{aligned}$$

Thus $S_v(u_0, 0)$ is also a bezier with control points $[S[1, 0] - S[0, 0], S[1, 1] - S[1, 0], \ldots, S[m, 1] - S[m, 0]]$.

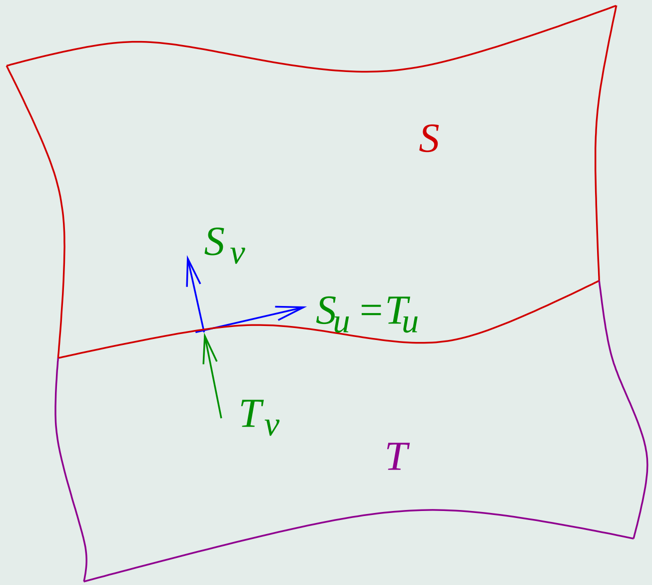
Pictorially



The normal at that point is given by the cross-product $S_v \times S_u$.

Splicing

Consider Two surfaces given by control points S and T. We would like to have them meet at a common boundary, and smoothly. Thus for example, we require $S(u,0) = T(u,1)$ for all $u \in [0,1]$. Furthermore, we require that the normals match too.



The conditions

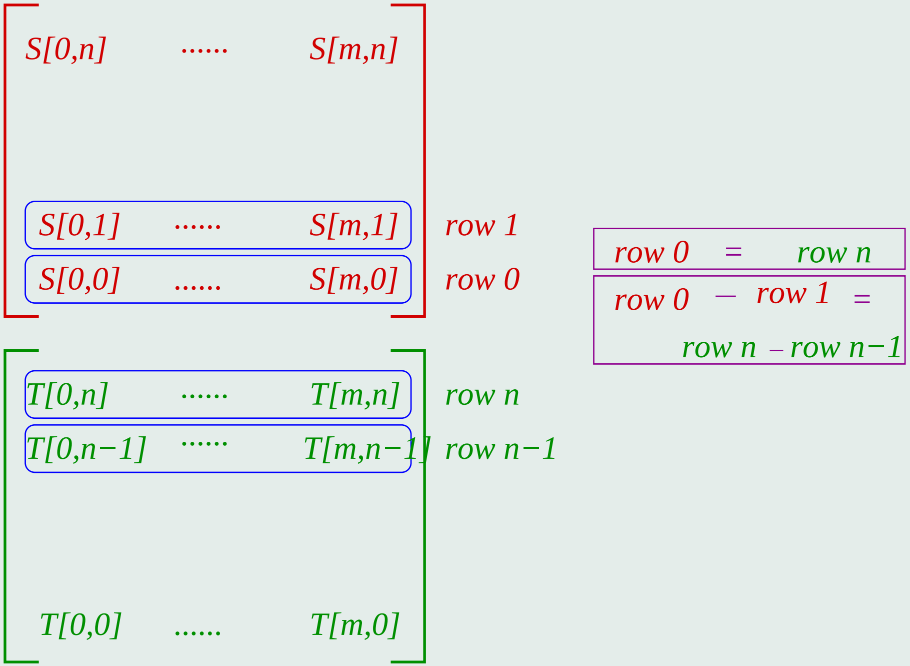
The condition $S(u, 0) = T(u, 1)$ is easily satisfied by having the bottom row of S match the top row of T.

This will also ensure that $S_u = T_u$ since both are tangents to the same curve.

Lets examine the normal condition next. $S_u \times S_v \equiv T_u \times T_v$, is achieved if we force S_v to be a multiple of T_v. This is forced by fixing a multiple, say α and requiring that:

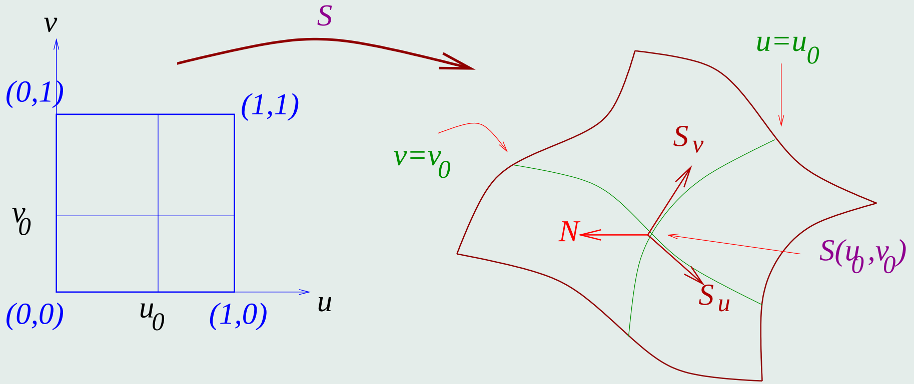
$$S[i, 1] - S[i, 0] = \alpha(T[i, n] - T[i, n - 1])$$

Schematically



The Normal System

A surface, if part of a solid, has at *every* point, an outward normal. Thus, given a (u_0, v_0) we are now faced with specifying *uniformly* an outward normal at $S(u_0, v_0)$!.



Consider the figure above. At the point $S(u_0, v_0)$, we have the two tangents S_u and S_v. Let $N = S_u \times S_v$. Clearly the outward normal at $S(u_0, v_0)$ must be either N or $-N$.

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The Sign of the Normal

We claim that if the outward normal at $S(u_0, v_0)$ is, say, $-N = -(S_u \times S_v)$, then it is so at every u, v [a].

Thus *all* that needs to be stored is a $sign \in \{+1, -1\}$. The normal at any point $S(u, v)$ is given by

$$sign \cdot (S_u \times S_v)$$

Proof: Let $U(u, v)$ be the unit outward normal which exists! Clearly, $U(u, v)$ is a smooth function on the surface.

Let $M(u, v) = sign \cdot \frac{S_u \times S_v}{|S_u \times S_v|}$. We see that (i) $M(u, v)$ is a smooth function on u, v, and (ii) $M(u, v)$ is normal at $S(u, v)$.

[a] provided $S_u \times S_v$ is never zero

Continued

Thus at all points (u, v), the vectors $M(u, v)$ and $U(u, v)$ are collinear. Now the proof goes in the following 3 steps:

- Since both are unit, we have $M(u, v)/U(u, v) \in \pm 1$.
- Since both U and M are smooth and unit, $M(u, v)/U(u, v)$ must be *uniformly* either $+1$ or -1.
- But we know that at (u_0, v_0) it is $+1$ and thus $M(u, v) = U(u, v)$.

$\square$

Things NOT covered

1. Surface Re-construction
2. Subdivision, Evaluation, Degree Elevation
3. Special Surfaces such as Coons-Patch
4. Tangent Planes, Gauss Map and Curvature