Bezier Curves

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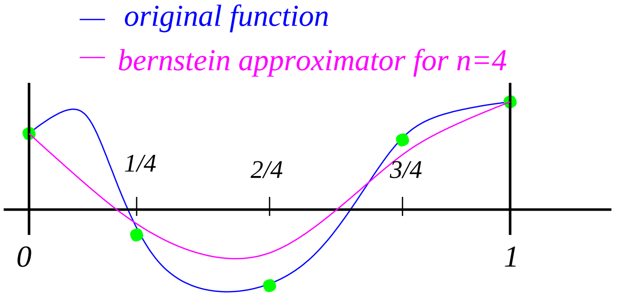
Recall

Lets recall a few things:

1. $f : [0,1] \rightarrow \mathbb{R}$ is a function.
2. $f_0, \ldots, f_i, \ldots, f_n$ are observations of f with $f_i = f(\frac{i}{n})$.
3. $B^n(f) = \sum_i f_i B_i^n(t)$ is a polynomial of degree n.
4. The plot of $B^n(f)l$ looks like this:



A Computation

- $\sum_{i=0}^{n} B_i^n(t) = 1.$

  This follows from binomial expansion of

  $$1 = ((1-t)+t)^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}$$

  Thus for all t, $B^n(f)(t)$ is a convex combination of the observations f_i.

- $\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t.$

  This is more delicate. Suppose we choose $f(t)$ as t itself, then $f(\frac{i}{n}) = \frac{i}{n}$. Thus what is being computed is the Bernstein approximation to $f(t) = t$. And what this says is that the approximation $B^n(f)$ is f itself!

  WARNING This is not true even for $f(t) = t^2$

Computation Continued...

We begin with the expression:

$$\begin{aligned} t &= \int 1.dt \\ &= \sum_{i=0}^{n-1} \int B_i^{n-1}(t)dt \end{aligned}$$

Now we solve this, and also eliminate the constant of integration. For this note that

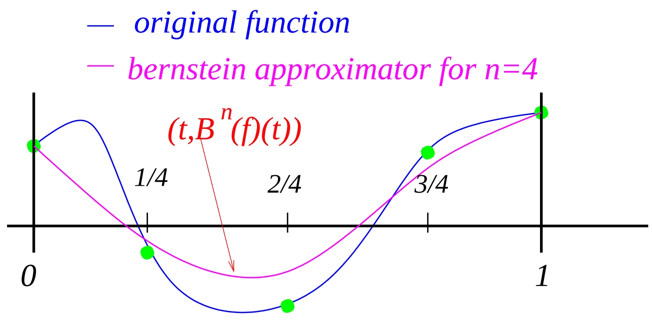
$$\int B_i^{n-1}(t)dt = \frac{1}{n} B_{i+1}^{n}(t) + \int B_{i+1}^{n-1}(t)dt$$

This easily telescopes into the desired result.

An Alternate Expression

Treating both $y = f(t)$ and $y = B^n(f)(t)$ as curves in $\mathbb{R}^2$, we can give a parametrization:

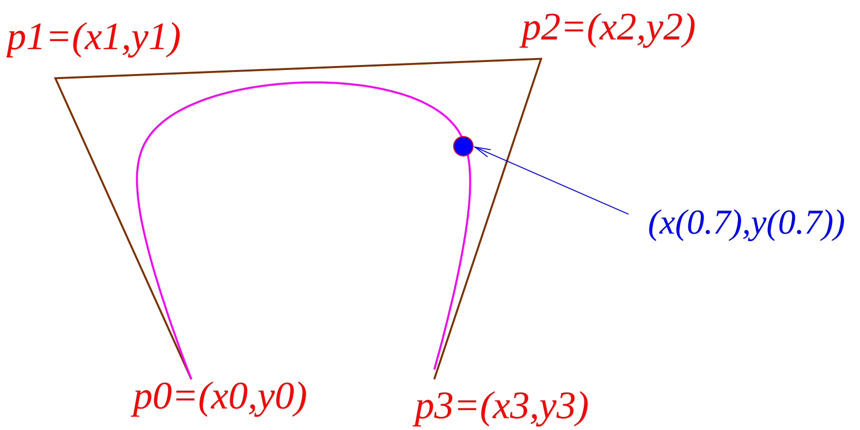
$$\begin{bmatrix} t \\ B^n(f)(t) \end{bmatrix} = \begin{bmatrix} \frac{0}{n} & \frac{1}{n} & \dots & \frac{n}{n} \\ f_0 & f_1 & \dots & f_n \end{bmatrix} \begin{bmatrix} B_0^n(t) \\ B_1^n(t) \\ \vdots \\ B_n^n(t) \end{bmatrix}$$



The Bezier Curve

In general, just as the y-coordinates were general, we may put general x-coordinates, instead of $\frac{i}{n}$ to get:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{bmatrix} \begin{bmatrix} B_0^n(t) \\ B_1^n(t) \\ \vdots \\ B_n^n(t) \end{bmatrix}$$



The Bezier Curve:Control Polygon

In general, if we have a sequence $P = [p_0, \ldots, p_n]$ of points $p_i = [x_i, y_i] \in \mathbb{R}^2$, we may define

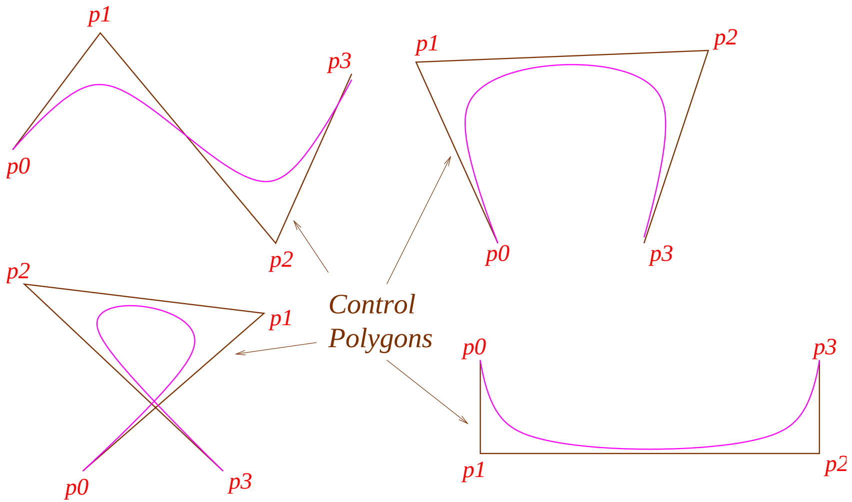
$$
\begin{aligned}
x(t) &= \textstyle\sum_{i=0}^{n} x_i B_i^n(t) \\
y(t) &= \textstyle\sum_{i=0}^{n} y_i B_i^n(t) \\
&\quad \text{or in general} \\
p(t) &= \textstyle\sum_{i=0}^{n} p_i B_i^n(t)
\end{aligned}
$$

$p(t)$ has nice properties such as $p(0) = p_0$, $p(1) = p_n$ and more.

The sequence $P = [p_0, \ldots, p_n]$ is called the control polygon.

A New Scheme

This gives us a new paradigm: Draw curves in space via the control polygon.



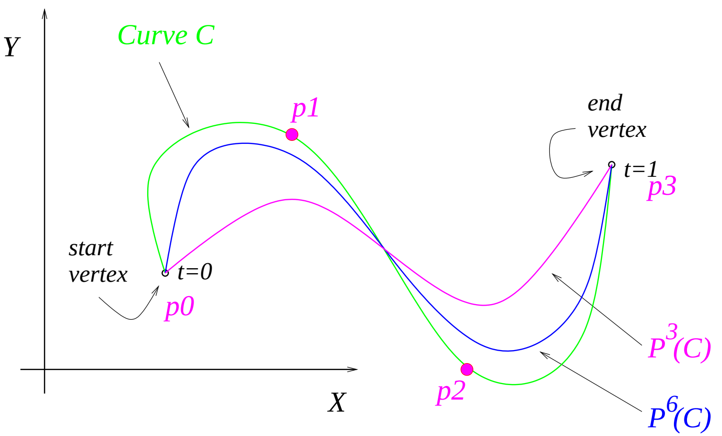
Bezier Curve: Using the Bernstein Basis and Control Polygons

The Construction of Given Curves

But what about approximation of already given curves?

- Given a curve C in $\mathbb{R}^3$, sample points $P^n = [p_0, \ldots, p_n]$ equi-distant along curve-length.
- Form $P^n(t) = \sum_i p_i B_i^n(t)$.

Theorem: For every $\epsilon > 0$, there is an n such that $P^n(t)$ is within the ϵ-envelop of C.



Bezier Curve Properties

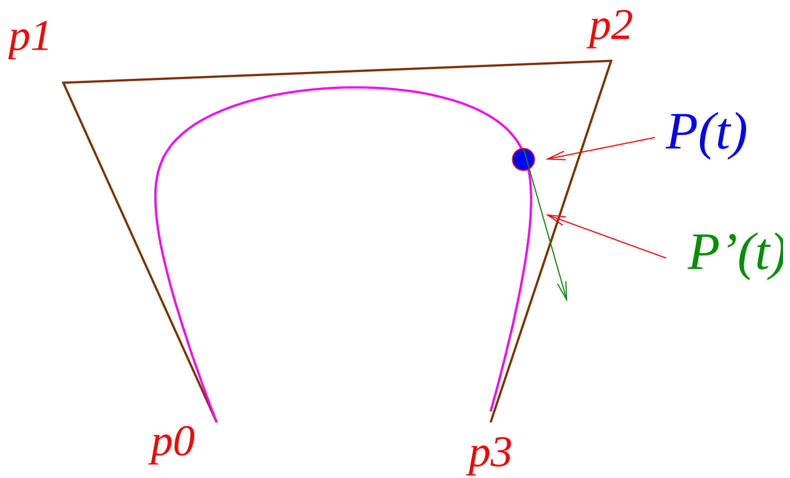
We begin with the expression:

$$P(t) = p_0 B_0^n(t) + p_1 B_1^n(t) + \ldots + p_n B_n^n(t)$$

- Putting $t = 0$, we see that B_i^n vanish for $i > 0$. and out pops p_0. Thus $\boxed{P(0) = p_0}$. Similarly $\boxed{P(1) = p_n}$. Thus the curve behaves quite predictably at the end-points.
- Next, for *any* $t \in [0, 1]$, we have $B_i^n(t) \geq 0$ and $\sum B_i^n(t) = 1$. Thus the curve $P(t)$ lies in the convex hull of the control polygon.

Tangents

So given $P = [p_0, \ldots, p_n]$, and $P(t) = \sum_i p_i B_i^n(t)$.
What is the meaning of $P'(t) = \frac{dP}{dt}$?
$P(t) = (x(t), y(t))$ and thus $P'(t) = (x'(t), y'(t))$ is the tangent to the curve.



Also recall that $\frac{dB_i^n(t)}{dt} = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$.

End Tangents

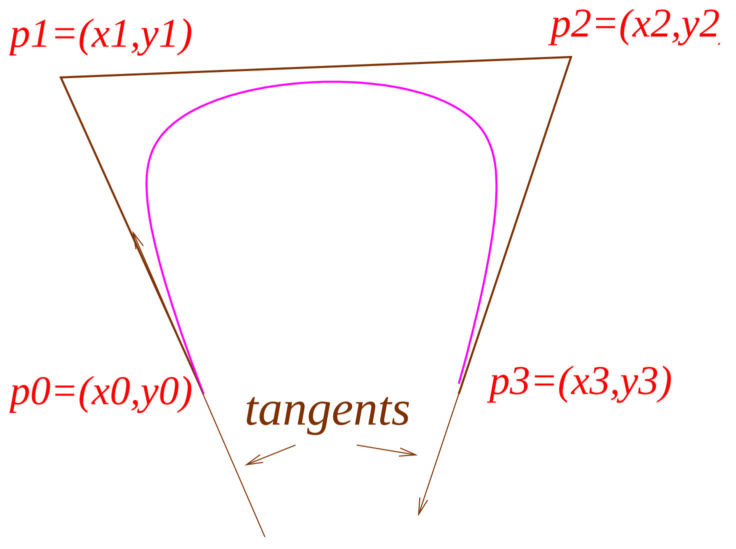
Back-substituting, we get that:

$$P'(t) = \sum_{i=0}^{n-1} q_i B_i^{n-1}(t) = \sum_{i=0}^{n-1} n(p_{i+1} - p_i) B_i^{n-1}(t)$$

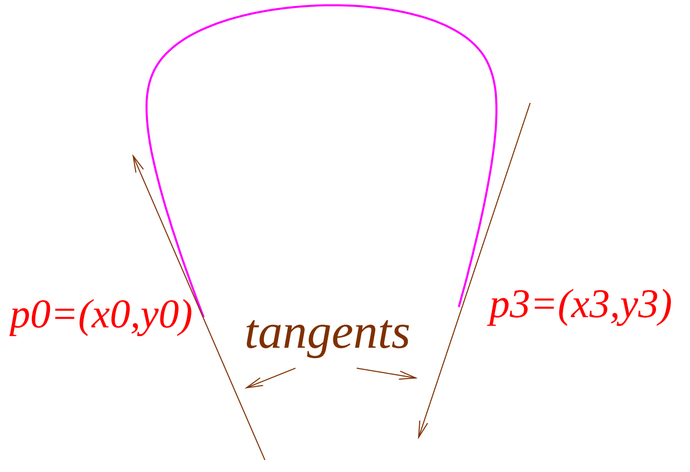
Thus, the derivative/tangent to $P(t)$ is a degree $n-1$ bezier curve, whose control points are easily computed.
Whence evaluating $P'(t)$ at 0, we see that $P'(0) = q_0 = n(p_1 - p_0)$, i.e.,

$$\begin{aligned} x'(0) &= n(x_1 - x_0) \\ y'(0) &= n(y_1 - y_0) \end{aligned}$$

Thus $P'(0)$, the tangent to the curve at 0 and is given by the line joining p_1 and p_0. The slope is clearly $\frac{y_1 - y_0}{x_1 - x_0}$.



Thus the behaviour of $P(t)$ at the end-points is easily determined from the control polygon: $P(0) = p_0$ and the tangent $P'(0) = (p_1 - p_0)/n$.



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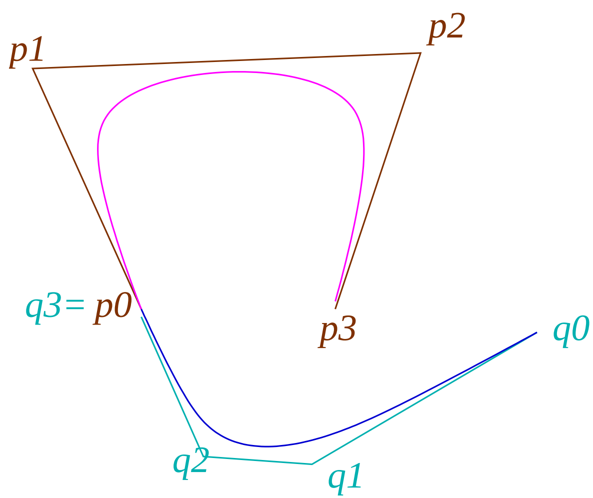
Caution: If we just know the image of $P(t)$, then p_0 is certainly determined as one of the end-points. From the tangent, we can just guess that p_1 lies on it.

Splicing

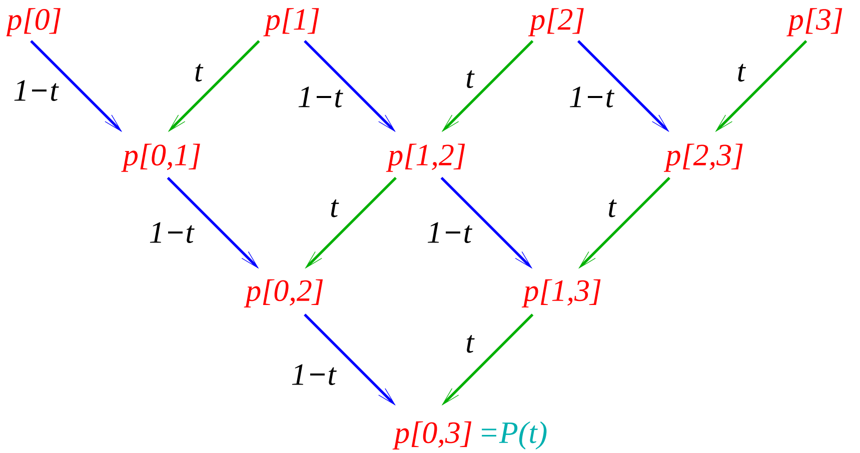
Question: Suppose P is a control polygon and $P(t)$ its associated curve. We would like to splice another curve $Q(t)$ which extends $P(t)$ at p_0. Then how is the control polygon of Q to be chosen?

Smooth extension result: The curve $Q(t)$ smoothly extends $P(t)$ if (i) $p_0 = q_m$ and (ii) $p_1 - p_0$ and $q_m - q_{m-1}$ are co-linear .



Evaluation: The deCasteljeu Algorithm

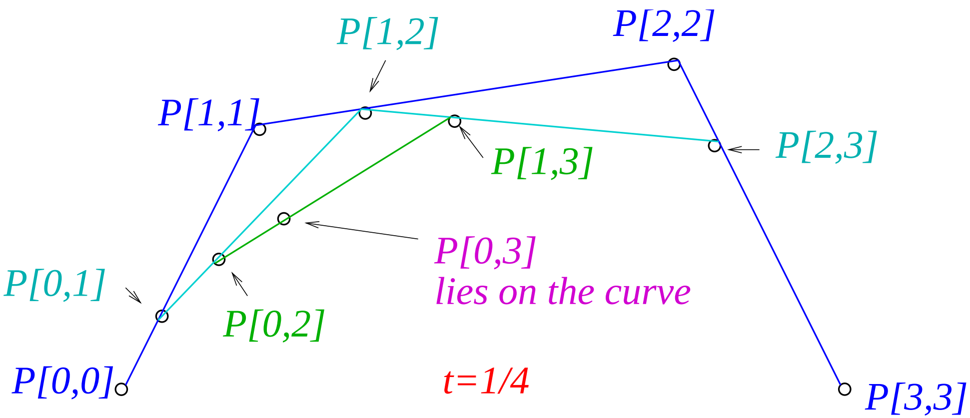
Question: How is one to evaluate $P(t)$, given $P = [p_0, \ldots, p_n]$ and the parameter value t.



The deCasteljeu scheme is $O(n^2)$, and quite efficient and stable.

Compare with evaluating $P(t) = \sum_{i=0}^{n} p_i \binom{n}{i} t^i (1-t)^{n-i}$ directly.

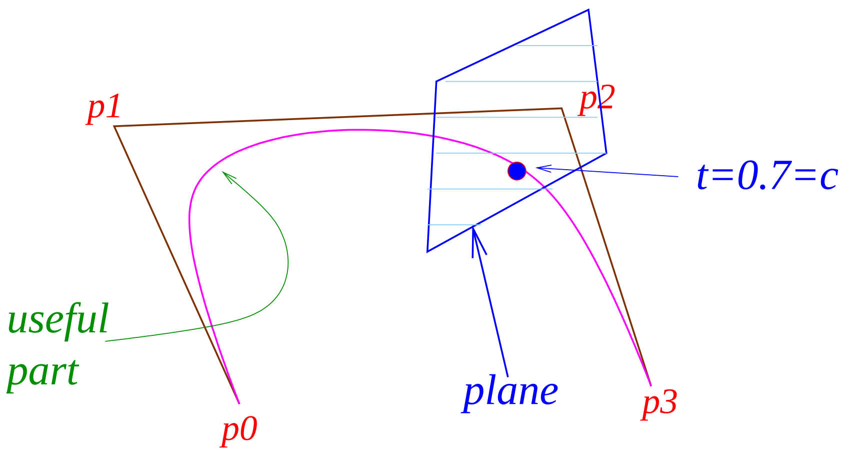
The Geometric De-Casteljeu



Thus every succesive iteration of the algorithm is a sequence of convex combinations of the points generated in the previous phase.
The final point $P[0n]$ thus is also (as expected) a convex combination of the elements of $P = [p_0, \ldots, p_n]$ and therefore lies in the convex hull of P.

Subdivision

Next consider the curve $C = P[t]$. Suppose that there is a surface S (a plane in this case) which intersects the curve C. Suppose that we have determined the intersection point and that it takes the parameter value $c = 0.7$.
The 'useful' part of the curve is C' which is C restricted to $t \in [0, 0.7]$.



Question: How is one to obtain the control points for C' having those of C?

In terms of polynomials...

Suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is a polynomial. For a given $c = 0.7$, we require another polynomial g such that $g(t) = f(ct)$.
Thus $g(0) = f(0)$ and $g(1) = f(c)$, and $g : [0,1] \longrightarrow \mathbb{R}$ defines the useful part of f.
If $f = a_0 + a_1 t^1 + \ldots + a_n t^n$, then

$$g(t) = f(ct) = a_0 + (c^1 a_1)t^1 + \ldots + (c^n a_n)t^n$$

In other words, $g = b_0 + b_1 t^1 + \ldots + b_n t^n$, where $b_i = c^i a_i$ for all i. Thus the expression of g in terms of the Taylor basis is clear when f is also similarly expressed.
So what happens when $f(ct) = \sum_{i=0}^{n} p_i B_i^n(ct)$, is expressed in the bernstein basis?

Subdivision in the Bernstein basis

In other words, express $B_i^n(ct)$ in terms of $\{B_0^n(t), \ldots, B_n^n(t)\}$. Trying our hand, we see that:

$$
\begin{aligned}
B_n^n(ct) &= \binom{n}{n}(ct)^n(1-ct)^{n-n} \\
&= c^n t^n = c^n B_n^n(t) \\
&= B_n^n(c)B_n^n(t) \\
B_{n-1}^n(ct) &= n(ct)^{n-1}(1-ct) \\
&= nc^{n-1}t^{n-1}[(1-t)+t(1-c)] \\
&= c^n B_{n-1}^n + nc^{n-1}(1-c)t^n \\
&= B_{n-1}^n(t)B_{n-1}^{n-1}(c) + B_n^n(t)B_{n-1}^n(c)
\end{aligned}
$$

In general, we have:

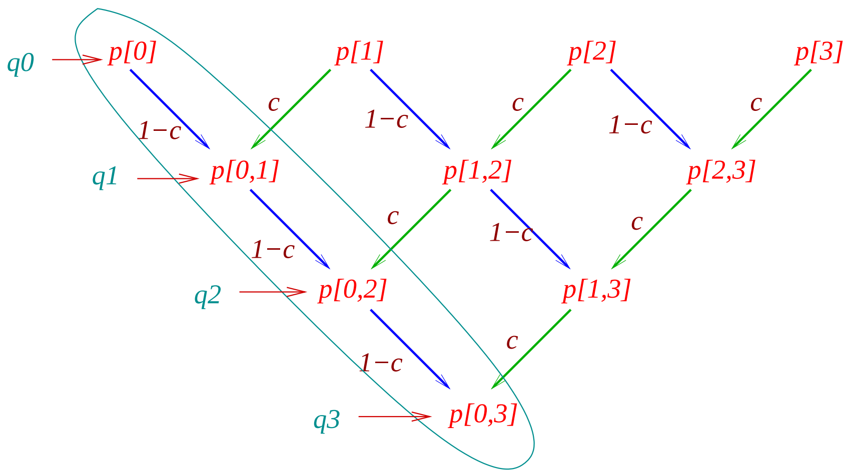
$$B_{n-k}^n(ct) = \sum_{j=0}^{k} B_{n-j}^n(t) B_{n-k}^{n-j}(c)$$

Back-Substituting...

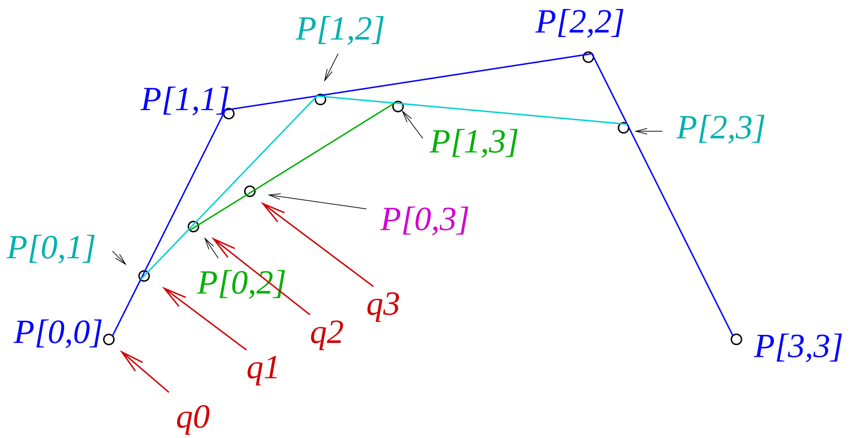
Back-substituting that expression into $P(ct) = \sum_{k=0}^{n} p_{n-k} B_{n-k}^{n}(ct)$, we get:

$$\begin{aligned} P(ct) &= \sum_{k=0}^{n} p_{n-k} \sum_{j=0}^{k} B_{n-j}^{n}(t) B_{n-k}^{n-j}(c) \\ &= \sum_{i=0}^{n} [\sum_{r=0}^{i} p_r B_r^i(c)] B_i^n(t) = \sum_{i=0}^{n} q_i(c) B_i^n(t) \end{aligned}$$

But we know this quantity before...:



And Geometrically speaking..



Thus, the control points for the sub-division are sitting there in the evaluation process.
Guess what are the control points for the second part of the curve, i.e., from $[c, 1]$?

Degree Elevation

Suppose $C = P[t]$ is given as a degree 3 parametrization. Thus $(x(t), y(t), z(t))$ are degree 3 polynomials. Then certainly, they are expressible in terms of $B_i^4(t)$! What is that expression?
In other words, given $P = [p_0, \ldots, p_n]$ compute $Q = [q_0, \ldots, q_n, q_{n+1}]$ so that:

$$\sum_{i=0}^{n} p_i B_i^n(t) = \sum_{j=0}^{n+1} q_j B_j^{n+1}(t)$$

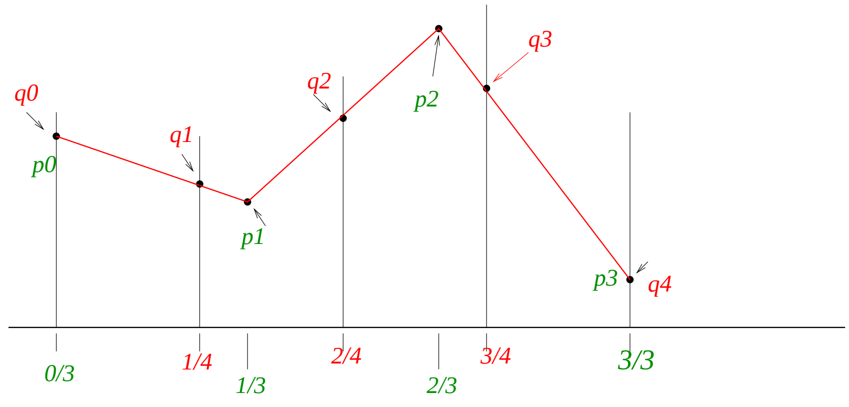
To begin with, we see:

$$\begin{aligned} B_i^n(t) &= \binom{n}{i} t^i (1-t)^{n-i} \\ &= (\binom{n}{i} t^i (1-t)^{n-i})[t + (1-t)] \\ &= \frac{i+1}{n+1} B_{i+1}^{n+1}(t) + \frac{n-i+1}{n+1} B_i^{n+1}(t) \end{aligned}$$

Back-substituting, we see that:

$$q_i = \frac{n-i+1}{n+1}p_i + \frac{i}{n+1}p_{i-1}$$

The coefficients have the following pleasing interpretation:



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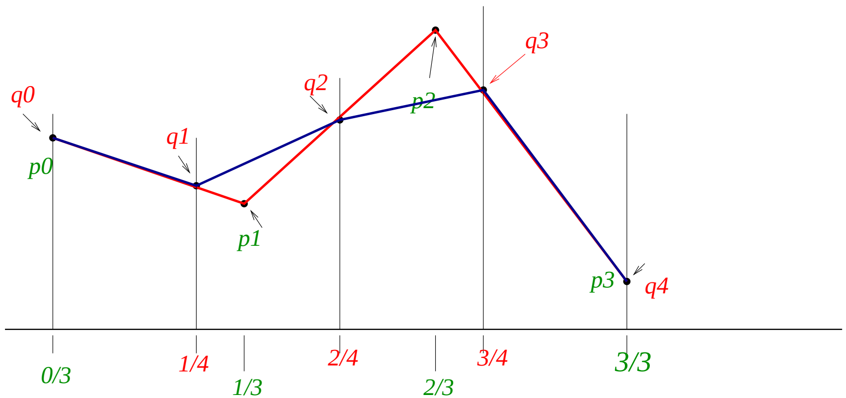

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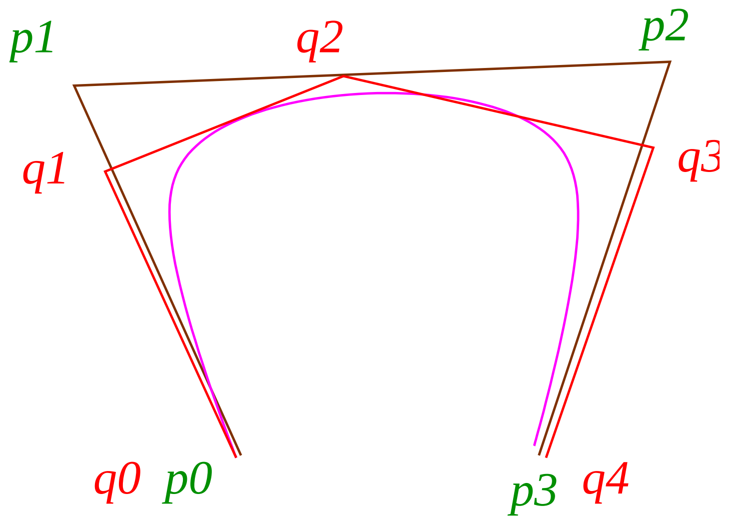

Thus, the control polygon Q may be obtained as an appropriate interpolation of the control-polygon P.

Wrap-Up

Recall that the edge geometry is stored as the tuple:

- an interval $[a, b]$, in this case $[0, 1]$.
- a map $f : [a, b] \rightarrow \mathbb{R}^3$, in this case, as a sequence $P = [p_0, \ldots, p_n]$.

Further:

1. The evaluation of f is given by the deCasteljeu algorithm. Also note that f is a polynomial and is thus defined beyond $[0, 1]$.
2. The subdivision and elevation are basic kernel operations of modifying the function f to suit requirements.
3. The construction of a particular curve from points on it is enabled via the Bezier-Bernstein theorem.