Polynomials and the Bernstein Base

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The Story So Far...

We have seen

- the 2-tier representation of faces/edges.
- parametrization as the choice of our representation
- within parametrization, the domain of definition and the function itself.

Recall that, for a curve, we had (i) $[a, b]$ an interval, and (ii) a function $x : [a, b] \rightarrow \mathbb{R}$, the X-coordinate of the curve parametrization. Similarly, $y, z : [a, b] \rightarrow \mathbb{R}$.
We shall now examine how to represent such functions.

Our Choice: Polynomials

The general polynomial is:

$$\boxed{p(t) = a_0 + a_1 t + \ldots + a_n t^n}$$

1. Ease of Representation-completely symbolic.
2. Ease of Evaluations-elementary operations.
3. Powerful theorems such as those of Taylor's, Lagrange interpolation and Bernstein Approximation.

The Polynomial Space

The general polynomial is

$$p(t) = a_0 + a_1 t + \ldots + a_n t^n$$

$P_n[t]$ will denote the space of polynomials of degree n or less. Note that $P_n[t]$ is a vector space, i.e.,

- It is closed under addition.
- It is closed under scalar multiplication

more ...

The dimension of $P_n[t]$ is $n+1$ and a basis for $P_n[t]$ is the Taylor basis

$$T_n = \{1, t, t^2, \ldots, t^n\}$$

In fact, $P_n[t]$ is isomorphic to $\mathbb{R}^{n+1}$ via this basis.

$$(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1} \Leftrightarrow a_0 + a_1 t^1 + \ldots + a_n t^n \in P_n[t]$$

Evaluation:

$$p(t) = a_0 + t[a_1 + t[a_2 + \ldots [a_{n-1} + t a_n]] \ldots]$$

Important: Different bases of $P_n[t]$ give different isomorphisms AND cater to different needs.

A Subtle Point

Supose we had chosen the class of *rational functions* as representation functions:

$$f_{a,b,c,d}(t) = \frac{at+b}{ct+d}$$

Thus we have 4 parameters and we may set up the map:

$$(a,b,c,d) \in \mathbb{R}^4 \Leftrightarrow f_{a,b,c,d}(t)$$

Then as functions is:

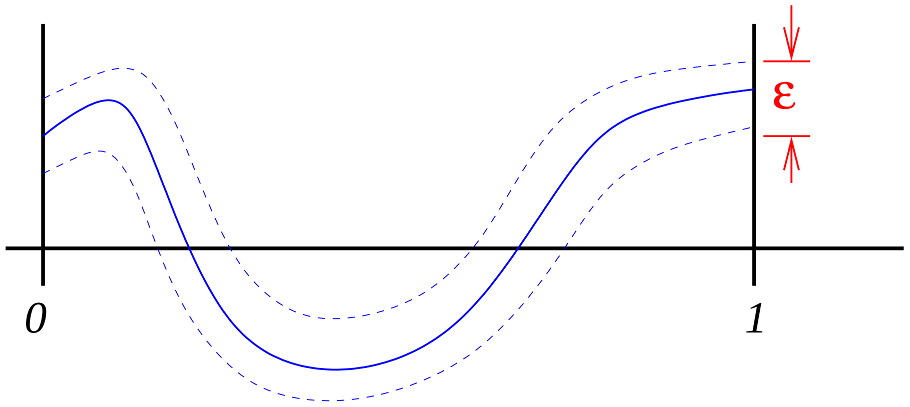
$$f_{a,b,c,d}(t) + f_{a',b',c',d'}(t) = f_{a+a',b+b',c+c',d+d'}(t)$$

The answer is NO.
Thus in the case of polynomials, the parameters $(a_0, \ldots, a_n)$ are indeed special!

$$\begin{array}{lcl} \text{Polynomials as functions} & \equiv & \text{Polynomials as coefficients} \\ \text{under addition} & & \text{under addition} \end{array}$$

Getting polynomials for functions

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a (coordinate) function.
Note that we may assume that $[a, b] = [0, 1]$ since polynomials are closed under translation.



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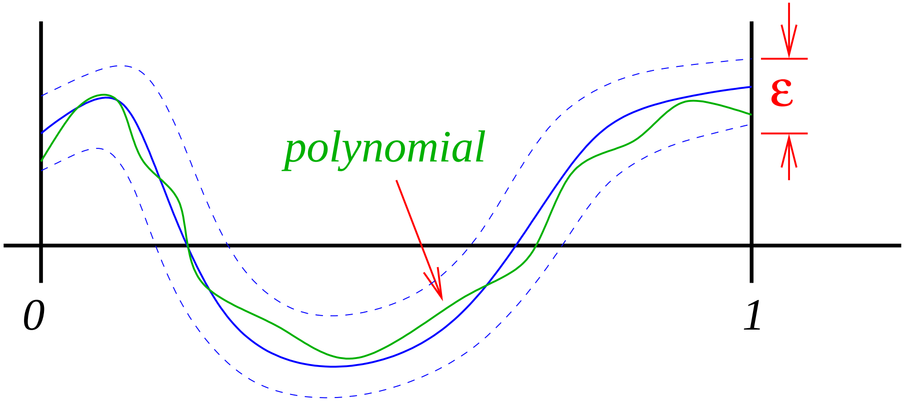

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The Taylor Approximation

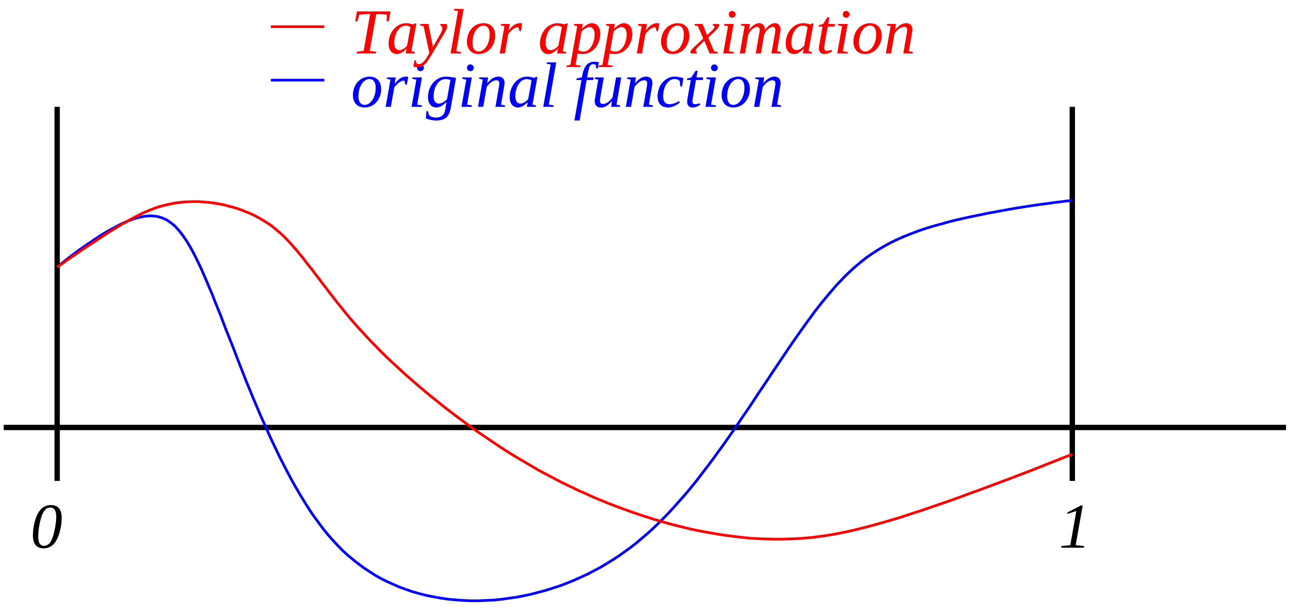
Let $f_0 = f(0), f_1 = f'(0), \ldots, f_n = f^n(0)$ be the $n+1$ derivatives at the point 0 and let $T_n(f)$ be the taylor approximation:

$$\boxed{T_n(f) = f_0 t^0 + \frac{f_1}{1!} t^1 + \ldots + \frac{f_n}{n!} t^n}$$

The function $T_n(f)(t)$ matches f at the point $t = 0$ and also the first n derivatives of f.

So how good is it?

Not too good...



.... in spite of $T_n(f)$ matching all derivatives at 0 with f.

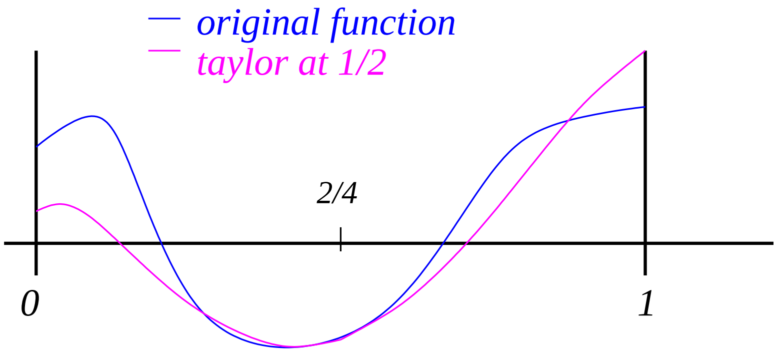
Another Taylor

Lets try

$$T_n^a = \{1, (t-a), (t-a)^2, \ldots, (t-a)^n\}$$

the taylor basis for the point $t = a$.

$$\boxed{T_n^a(f) = f(a)t^0 + \frac{f^1(a)}{1!}t^1 + \ldots + \frac{f^n(a)}{n!}t^n}$$



What about Interpolation at many points?

The Lagrange Basis.: Let $t_0, \ldots, t_n$ be $n+1$ distinct **points of observation**. Let

$$\boxed{L_i(t) = \frac{\prod_{j \neq i}(t - t_j)}{\prod_{j \neq i}(t_i - t_j)}}$$

Note that $L_i(t_j) = 0$ is $i \neq j$ and 1 otherwise.
Use: Let $f(t_i) = f_i$ and let

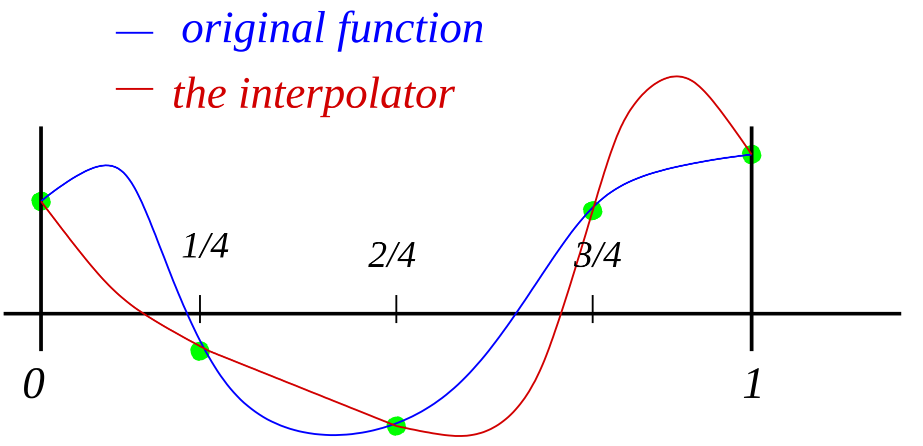
$$L^n(f) = \sum_{i=0}^{n} f_i L_i(t)$$

Note that

$$\boxed{L^n(t_i) = f(t_i) = f_i \text{ for all } i}$$

So lets plot $L^n(f)$

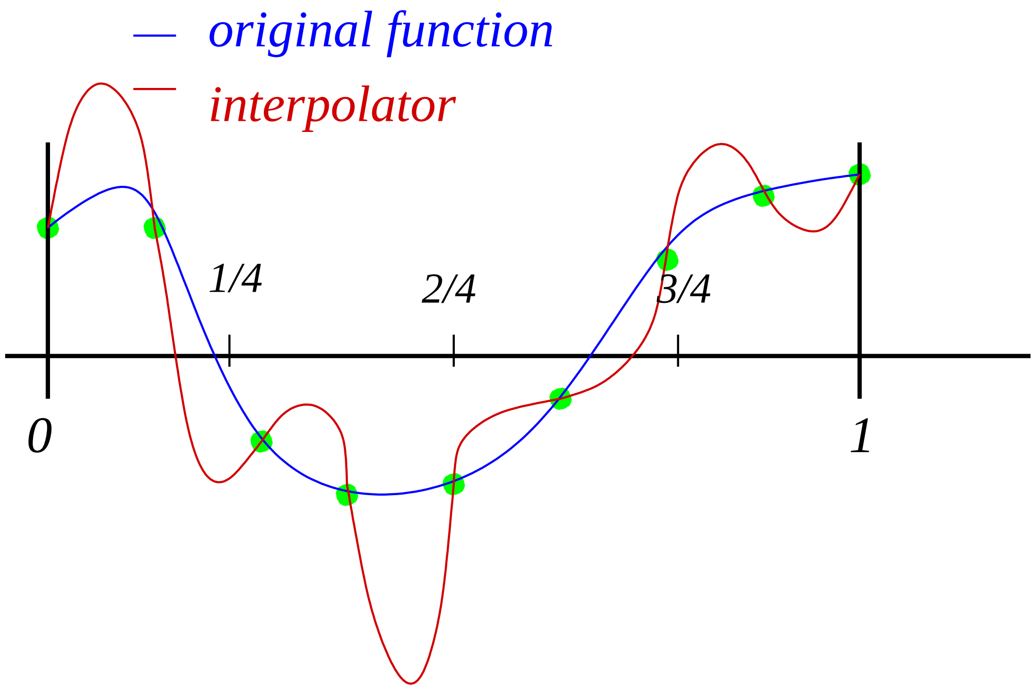
We get ....



Thats bad. Again, inspite of $L^n(f)$ matching f at $t = 0, 1/4, \ldots, 4/4$.

Perhaps more interpolation points will help....

And we get....



In fact, in general the interpolator is usually never an approximator. The closer the interpolation points, the wider the swings.

The Bernstein Basis[a]

$$\boxed{B_i^n = \binom{n}{i} t^i (1-t)^{n-i}}$$

Define for $i = 0, 1, \ldots, n$, the observation at $n+1$ equally spaced points:

$$\boxed{f_i = f(\tfrac{i}{n})}$$

Form the n-th bernstein approximant:

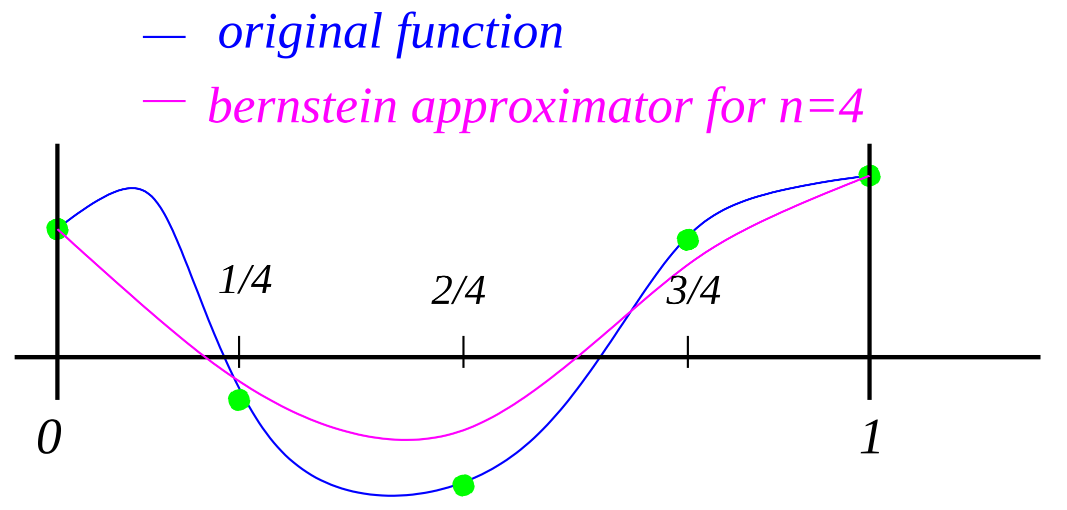
$$\boxed{B^n(f) = \sum_{i=0}^{n} f_i B_i^n(t)}$$

[a] Verify that this indeed a basis of $P_n[t]$

Thus for $n = 4$ we have the observations $f(0)$, $f(1/4)$, $f(2/4)$, $f(3/4)$ and $f(4/4)$. We get the degree 4 polynomial:

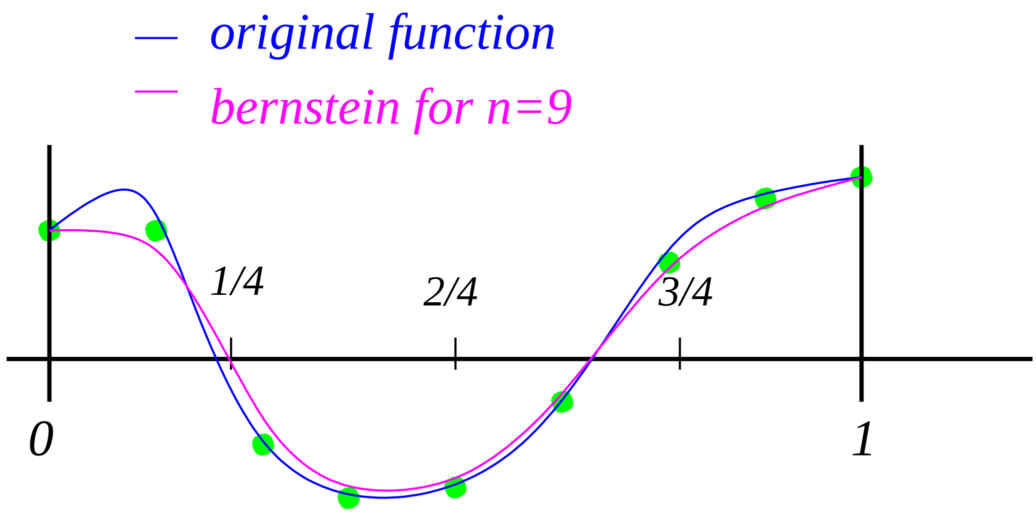
$$\boxed{B^4(f) = \sum_{i=0}^{4} f_i B_i^4(t)}$$

On plotting it, we see:



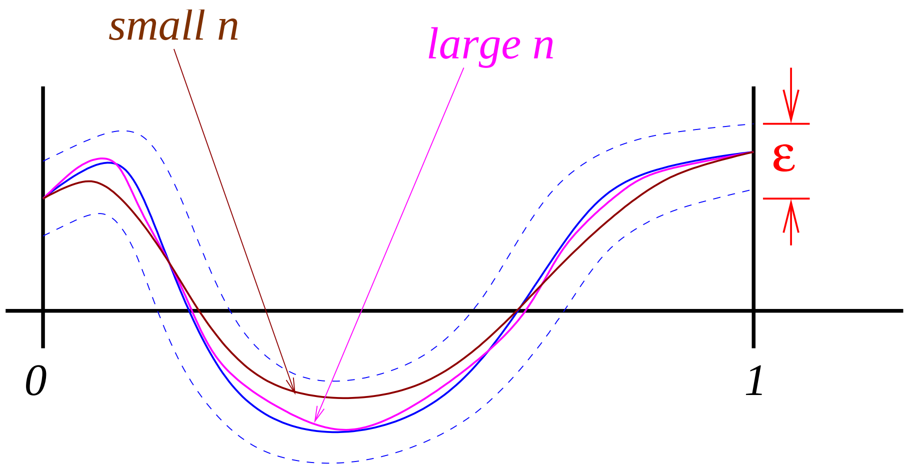
Things get better...

With $n = 9$ and 10 equally spaced observations, we have:



The Bernstein-Weierstrass Theorem

If $f : [0,1] \to \mathbb{R}$ is a continuous function, and $\epsilon > 0$, then there is an n such that $B^n(f)$ approximates f on $[0,1]$ within ϵ.

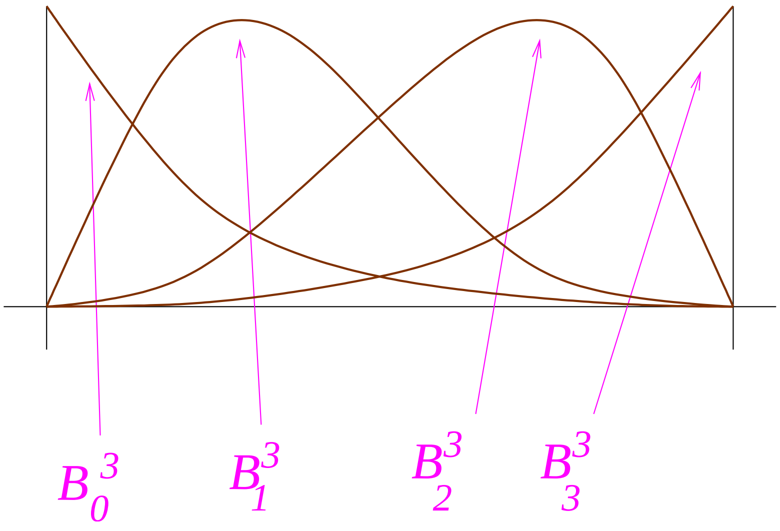


Thus there is a systematic way of getting better and better approximations.

Bernstein Polynomials

$$\boxed{B_i^n = \binom{n}{i} t^i (1-t)^{n-i}}$$

- $B_i^n(0) = 0$ unless $i = 0$, in which case $B_0^n(0) = 1$.
- $B_i^n(1) = 0$ unless $i = n$, in which case $B_n^n(1) = 1$.
- $B_i^n(t) \geq 0$ for $t \in [0, 1]$.



More properties

$$\boxed{B_i^n = \binom{n}{i} t^i (1-t)^{n-i}}$$

- $\int_0^1 B_i^n(t)dt = \frac{1}{n+1}$.
- $\frac{dB_i^n(t)}{dt} = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$
- The maximum value of $B_i^n(t)$ occurs at the point $\frac{i}{n}$.

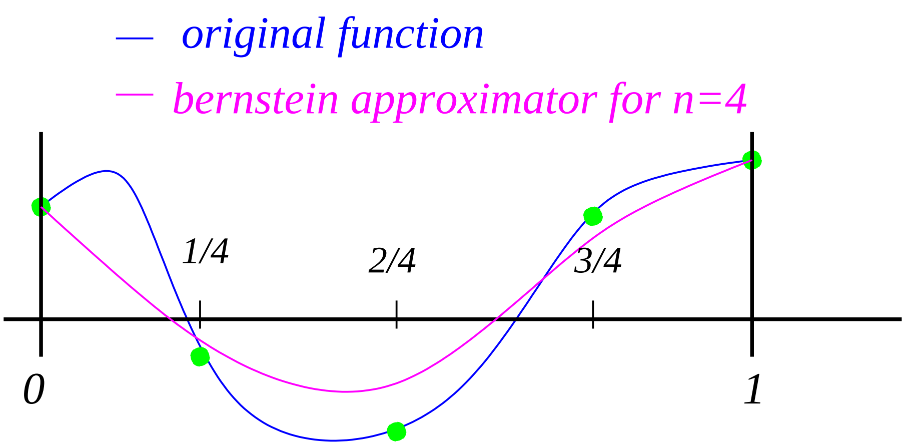
We just prove one of them:

$$\begin{aligned} \frac{dB_i^n(t)}{dt} &= i\binom{n}{i} t^{i-1}(1-t)^{n-i} - (n-i)\binom{n}{i} t^i (1-t)^{n-i-1} \\ &= n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)) \end{aligned}$$

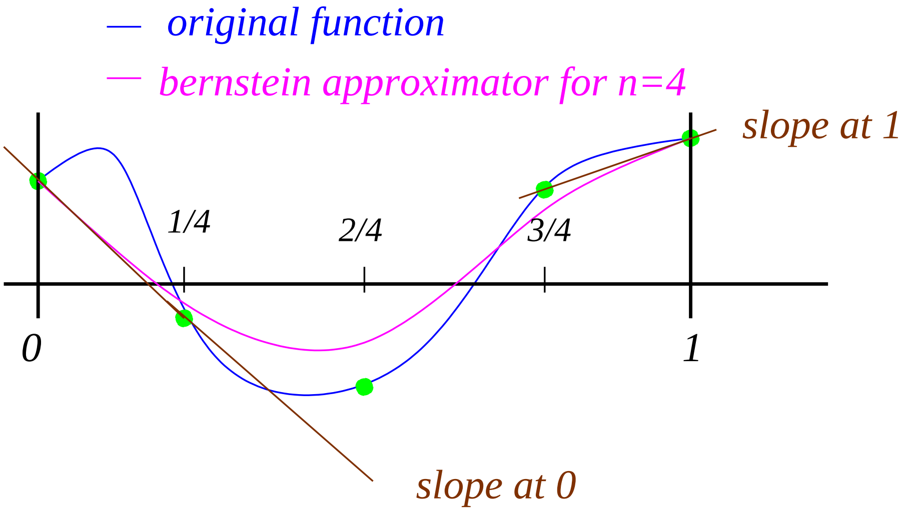
Properties of $B^n(f)$

$B^n(f) = \sum_{i=0}^{n} f_i B_i^n(t)$ $\qquad$ $B_i^n = \binom{n}{i} t^i (1-t)^{n-i}$

- $B^n(f)(0) = f(0)$ and $B^n(f)(1) = f(1)$.
  After all $B_i^n(0) = 0$ unless $i = 0$. Thus the only term is $f_0 = f(0)$.
  Caution: $B^n(f)(i/n) \neq f(i/n)$.



- $\frac{dB^n(f)}{dt}(0) = \frac{f(1/n)-f(0)}{1/n}$.
- $\int_0^1 B^n(f)(t)dt = \sum_{i=0}^{n} \frac{1}{n+1} \cdot f(i/n)$.



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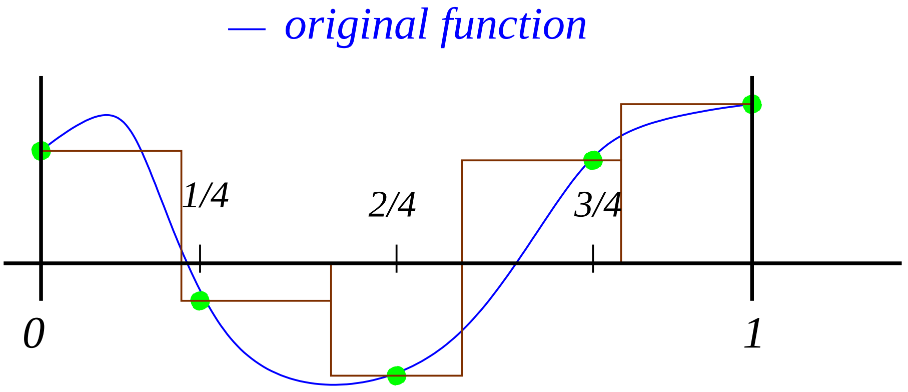
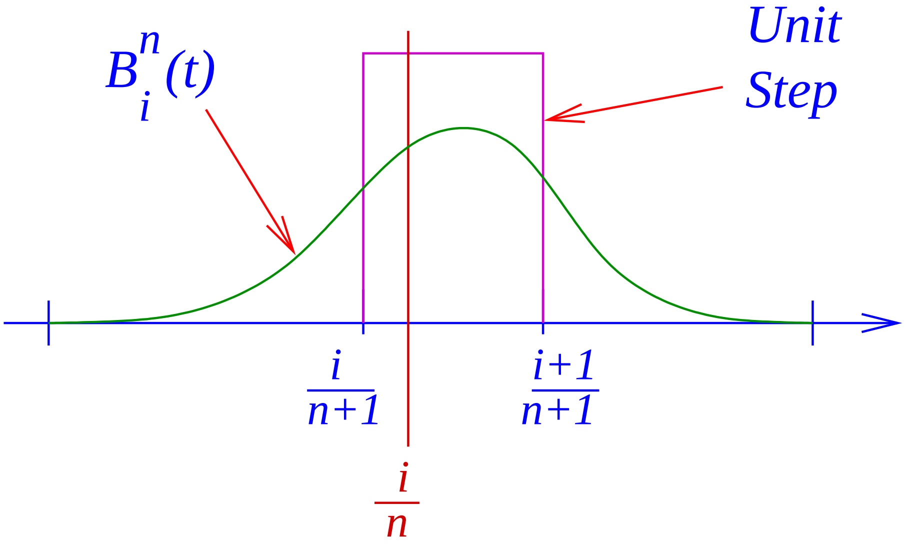


Thus, In A Way..

The function $B_i^n(t)$ behaves like the unit-step function for the interval $[\frac{i}{n+1}, \frac{i+1}{n+1}]$.



Also note that the *observation point* $\frac{i}{n}$ belongs to the above interval.

A pause

In general, we have had $n+1$ linearly independent observations, and a basis to match them.

Taylor	$f(0), f'(0), f''(0), f'''(0)$
Lagrange	$f(0), f(1/4), f(2/4), f(1)$
Bernstein	approximate everywhere! based on Lagrange data
Hermite	$f(0), f'(0), f(1), f'(1)$