B-Splines

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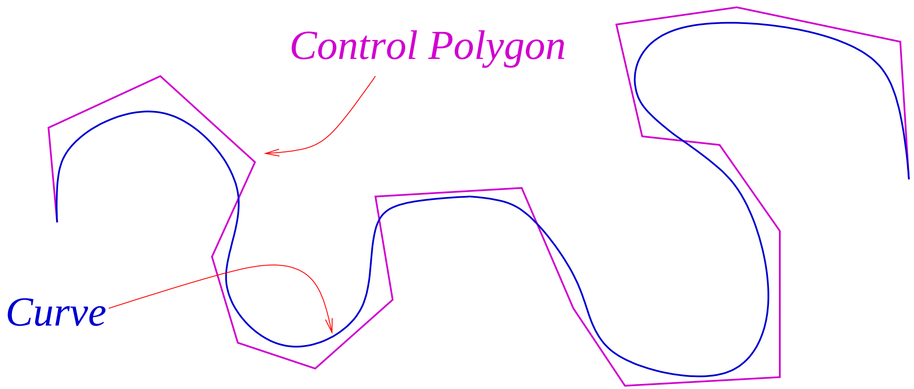
An Issue

Suppose we are to model a long curve with many convolutions. How does the bezier paradigm do?

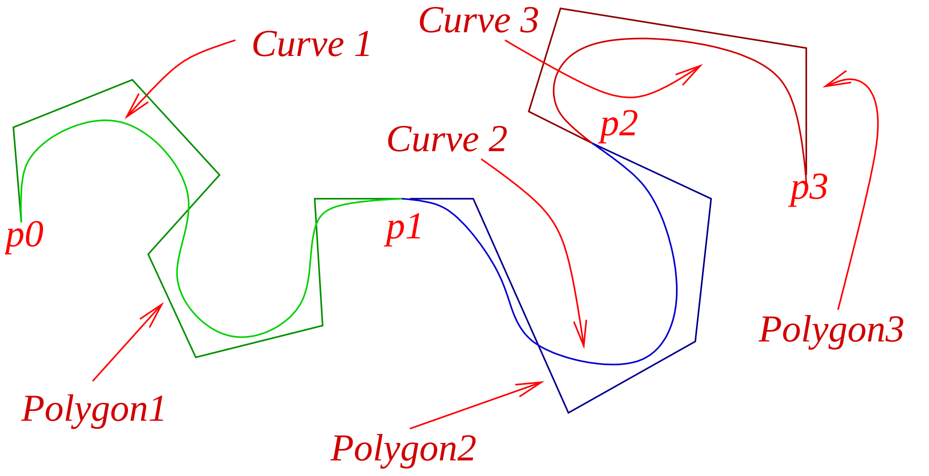
Option1: Use as many control points as required to model the curve:



Problem with this is that as the number of control points increase, the time to evaluation, which is $O(n^2)$ increases as a square in this quantity. This can be very expensive.

Option 2

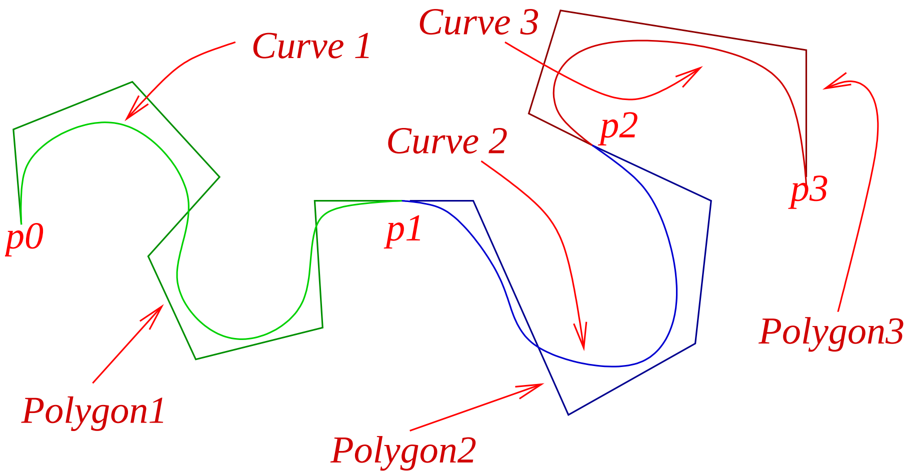
Option 2: Break up the curve into many parts and model separately.



This is a good option except that continuity at the junction points $p1, p_2, p_3$ and p_4 poses some problems.

C^0-continuity is easy to impose; just make sure that the last control point of C_1 equals the first of C_2.

Higher Continuities?



- C^1-continuity is a bit more tedious: the last span of the control polygon P_1 should be colinear with the first of P_2.
- C^2-continuity is even more tedious.

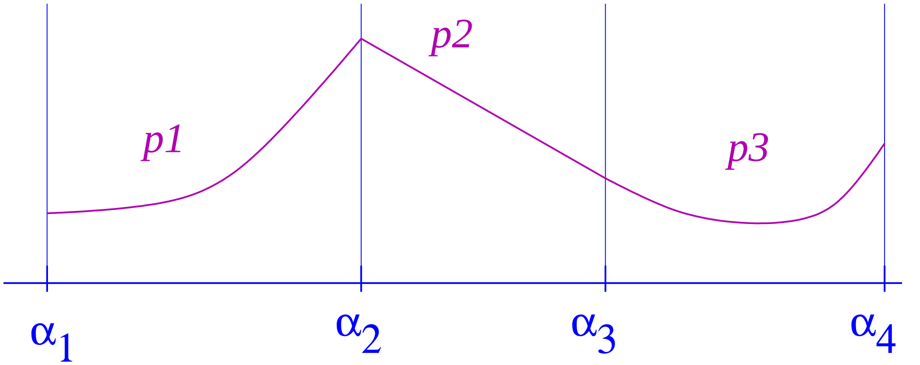
So if this jugglery can be managed, then Option 2 is acceptable.

Piece-wise Polynomials

Fix

- a degree D.
- a sequence $\alpha_1 < \alpha_2 < \ldots < \alpha_k$ of real numbers.

A function $f : [\alpha_1, \alpha_k] \longrightarrow \mathbb{R}$ is a piece-wise polynomial for the above data if there are polynomials $p_1(t), \ldots, p_{k-1}(t)$ of degree atmost D such that $f(t) = p_i(t)$ whenever $t \in [\alpha_i, \alpha_{i+1}]$.



Notice that f appears to be C^0-continuous at α_2 and C^1-continuous at α_2.

Defect

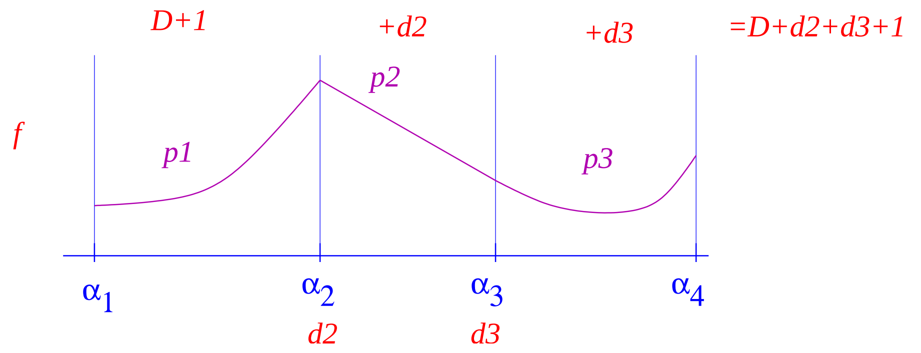
Question: What is the maximum k so that p_1 and p_2 are C^k-continuous at α_2?
Answer: Obviously the degree D, in which case p_1 and p_2 are identical
Indeed, the $D+1$ relations that $p_1(\alpha_2) = p_2(\alpha_2)$ and $p_1'(\alpha_2) = p_2'(\alpha_2)$ and so on till $P_1^D(\alpha_1) = p_2^D(\alpha_2)$ enforce that $p_1 = p_2$.
Let $f = (p_1, \ldots, p_{k-1})$ be a piece-polynomial. We say that f has a defect of (atmost) d at α_1 if:

$$p_1^i(\alpha_2) = p_2^i(\alpha_2) \text{ for } i = 0, 1, \ldots, D - d.$$

Free Dimensions

Thus if p_1 is known and the defect at α_2 is d_2 then there are exactly d_2 more conditions needed to define p_2 completely. Carrying on like this, we see that, roughly speaking, the degrees of freedom for a piece-wise polynomial function f is

$$D + 1 + d_2 + d_3 + \ldots + d_{k-1}$$



Knot Vector

The data D, $(\alpha_1, \ldots, \alpha_k)$ and the prescribed maximum defects $d_2, \ldots, d_{k-1}$ are succintly expressed in the format of a knot vector

Knot Vector: $\overline{\beta} = [\beta_1 \leq \beta_2 \leq \ldots \leq \beta_m]$ such that:

- No entry occurs more than D times.
- $\beta_1 = \beta_2 = \ldots = \beta_D$ and $\beta_{m-D+1} = \beta_{m-D+2} = \ldots + \beta_m$.

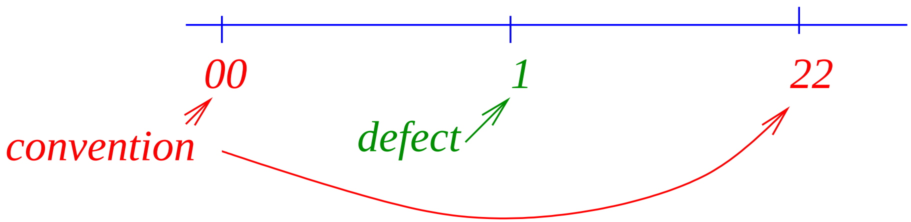
D is called the degree of the knot-vector, m its length. For a $\beta \in \overline{\beta}$, the multiplicity of β is the number of occurrences of β in $\overline{\beta}$.

Examples:

- $S = [0, 0, 0, 1, 1, 1]$: This is the standard *bezier* knot vector of degree 3.
- $O = [0, 0, 0, 2, 4, 4, 4]$, degree 3 and length 7.
- $A = [0, 0, 0, 2, 2, 4, 4, 4]$, degree 3 and length 8.
- $B = [0, 0, 0, 1, 2, 2, 4, 4, 4]$, degree 3 and length 9.
- $D = [0, 0, 0, 1, 2, 2, 3, 4, 4, 4]$, degree 3 and length 10.

Interpretation: A Small Example

Lets look at $[00122]$.
$V([00122])$ will denote the space of all piece-wise polynomial functions of degree 2 on $[0,2]$ with defect 1 at 1.

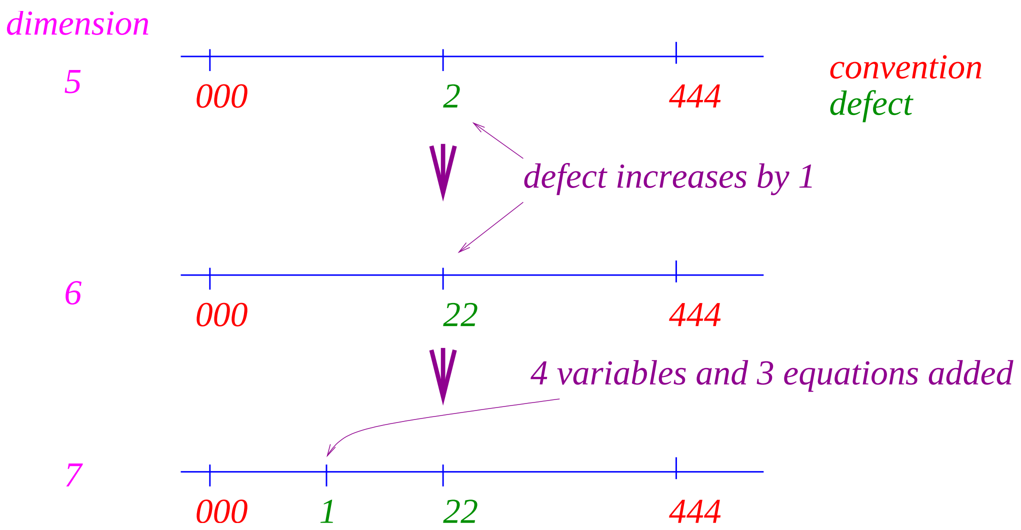


Thus V consists of two degree 2 polynomials p_1, p_2 such that:
(i) $p_1^1(1) = p_2^1(1)$ (ii) $p_1^0(1) = p_2^0(1)$
Since $p_1 = a_0 + a_1 t + a_2 t^2$, and $p_2 = b_0 + b_1 t + b_2 t^2$, we have 6 variables and 2 relations between these variables. The relations are:

$$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 \qquad \text{and} \qquad 2a_2 + a_1 = 2b_2 + b_1$$

Thus dimension of V is 4.

Pictorially, a larger example



Thus by an insertion of a **knot**, the dimension increases by exactly 1. It is easy to show now that:

$$dim(V(A)) = length(V(A)) - D + 1$$

Interpretation: More Examples

- $V(S) = V[000111]$ is space of all cubic polynomial functions on $[0, 1]$.
- $V(O) = V[0002444]$ is the space of all piece-wise polynomial functions on $[0, 4]$ with defect 1 at 2.
- $V(A) = V[00022444]$ is the space of all piece-wise polynomial functions on $[0, 4]$ with defect 2 at 2.
- $V(B) = V[000122444]$ is the space of all piece-wise polynomial functions on $[0, 4]$ with (i) defect 1 at 1 and (ii) defect 2 at 2.

Note that $V(O) \to V(A) \to V(B)$.

$dim(V(S)$	4
$dim(V(O)$	5
$dim(V(A)$	6
$dim(V(B)$	7

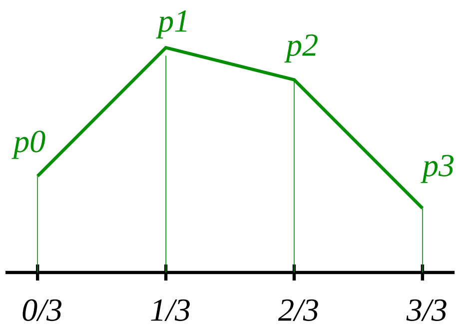
Greville Abscissa

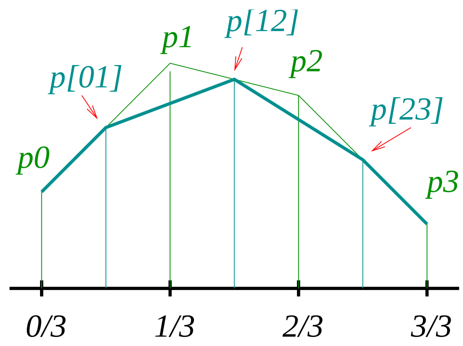
Question: But what about a basis for $V(A)$?
Recall the bezier case. We had:

- Special points $\xi_i = \frac{i}{n}$ and $Gr_n = \{\xi_0, \xi_1, \ldots, \xi_n\}$.
- A basis element $B_i^n(t)$ associated with each point ξ_i.
- Control polygon $P = [p_0, \ldots, p_n]$ with p_i associated with each ξ_i.
- An evaluation procedure based on interpolation within this polygon.

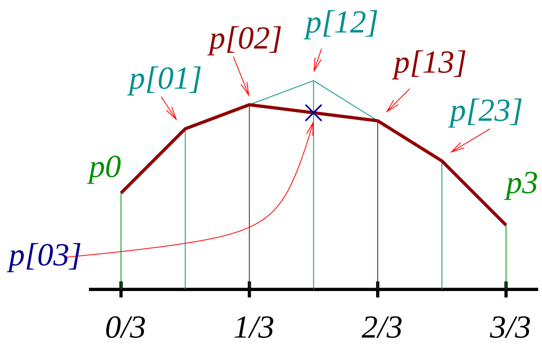
A similar process happens for general knot vectors.

Re-Cap





The deCasteljeu procedure



The General Case

We pick the knot vector) $=[0002444]$. We define the set $Gr(A)$ to be averages of D consecutive knots in the knot vector.
Thus $Gr(A)=\{0,2/3,2,10/3,4\}$. Note that

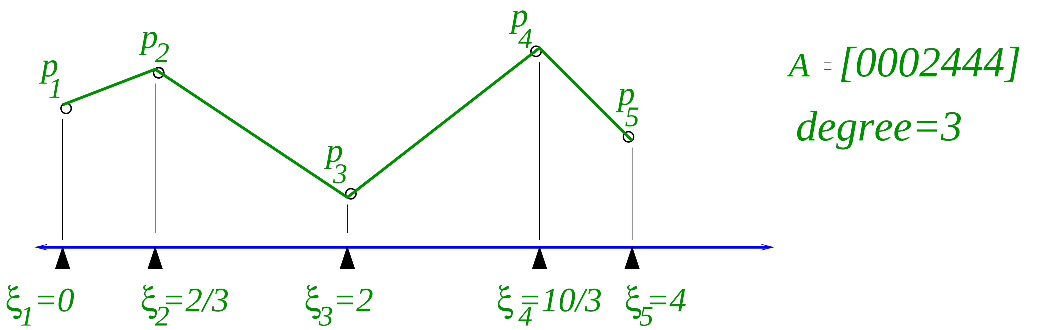
- $|Gr(A)| = length(A) - D + 1$.
- The first and the last knot are elements of $Gr(A)$. In fact if a knot has multiplicity D then it shows up as a greville abscissa.
- $Gr([000111]) = \{0/3, 1/3, 2/3, 3/3\}$.

Formally, $\overline{\beta} = \beta_1, \ldots, \beta_m$ is the knot vector then $Gr(\beta) = \{\xi_1, \ldots, \xi_{m-D+1}\}$, where

$$\xi_i = \frac{\beta_i + \beta_{i+1} + \ldots + \beta_{i+D-1}}{D}$$

The Control Polygon

Assign to each element ξ_i of $Gr(\beta)$, a control point. Locate the set $Gr(\beta)$ on the real line and form the Control Polygon.



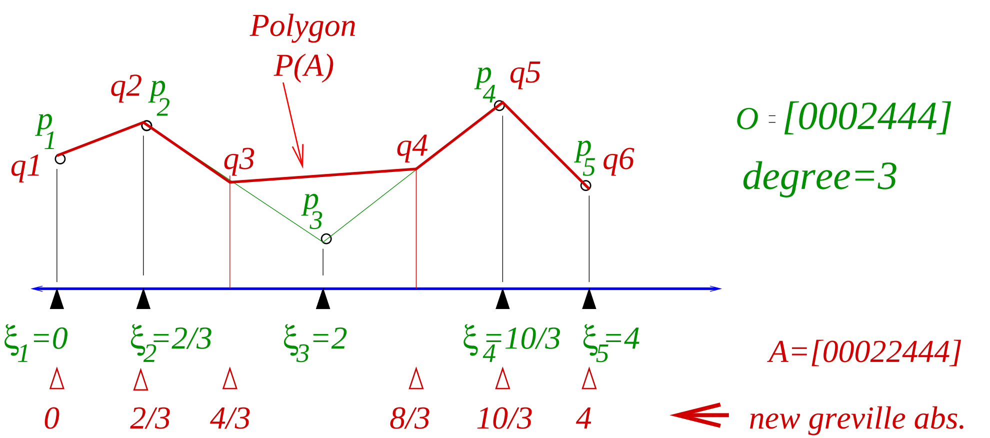
The Knot Insertion

The basic process is knot insertion. Suppose that we are given a polygon $P = P(O)$ on $Gr(O)$. And suppose that A is obtained from O by inserting a knot in O. We construct a polygomn $Q = P(A)$ on $Gr(A)$ from $P(O)$ as follows:

- Compute $Gr(A)$. This set has one more element than $Gr(O)$. In fact, each element of $Gr(A)$ lies between two elements of $Gr(O)$ or is equal to one of them.
- For each $\eta \in Gr(A)$, express η as a convex combination of two adjacent elements ξ_i and ξ_{i+1} of $Gr(O)$.
- Use these coefficients to obtain $Q(\eta)$ as a convex combination of $P(\xi_i)$ and $P(\xi_{i+1})$.

This is shown in the next slide.

Pictorially



We see that

$$\begin{aligned} 4/3 &= 1/2 \cdot 2/3 + 1/2 \cdot 2 \\ 8/3 &= 1/2 \cdot 2 + 1/2 \cdot 10/3 \\ &\text{thus} \\ q_3 &= 1/2 \cdot p_2 + 1/2 \cdot p_3 \\ q_4 &= 1/2 \cdot p_3 + 1/2 \cdot p_4 \end{aligned}$$

Evaluation

Inputs:

1. The knot vector A.
2. The control points (polygon) on $Gr(A)$.
3. The parameter t.

Output: $f(t)$.

1. Compute $Gr(A)$ and store it.
2. Insert t into A D times or till the multiplicity of t becomes D.
   - Add t and re-compute greville abscissa.
   - Intrpolate to get the new control polygon.
3. Now t is a greville abscissa. Read off the value at t as $f(t)$.

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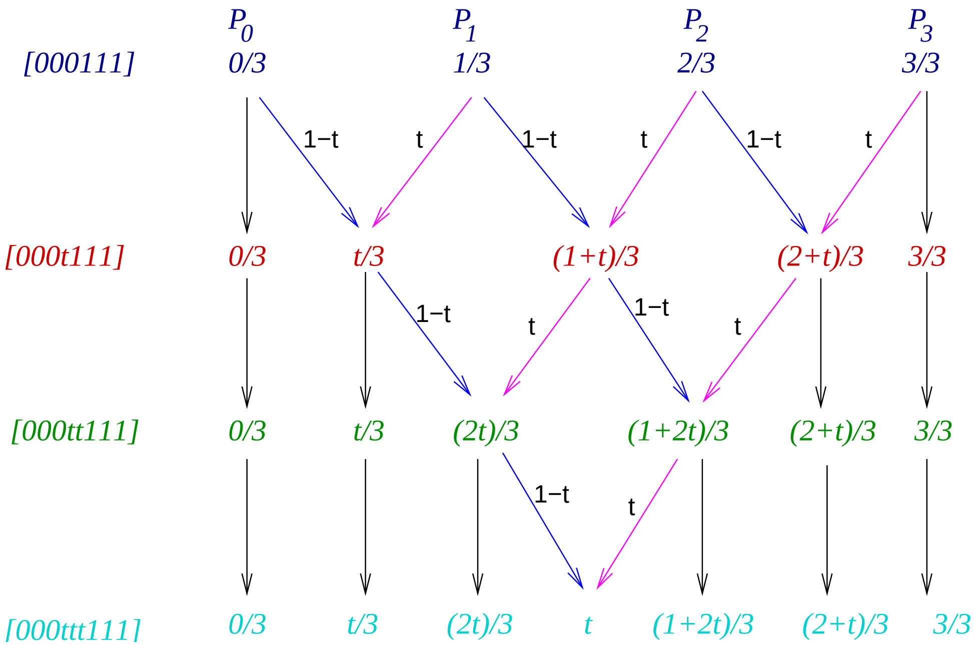
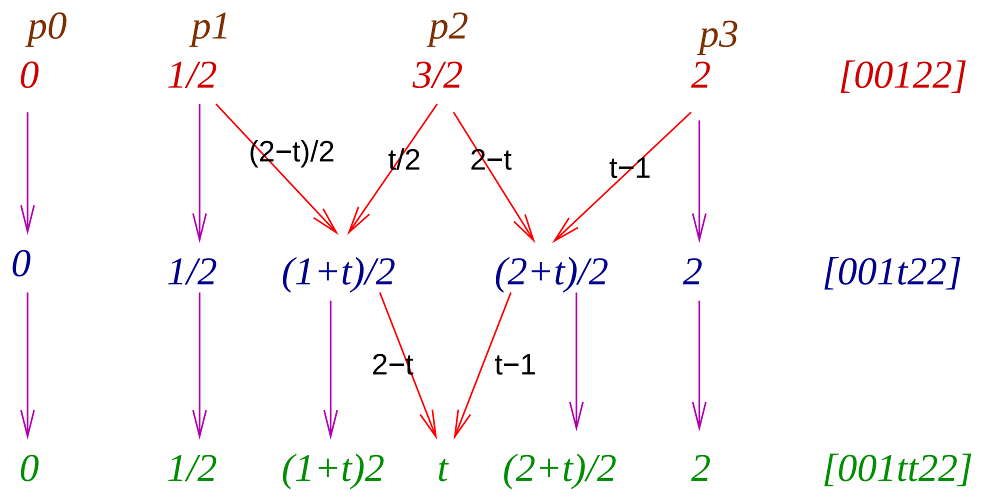
The Bezier Case



This is just the de-Casteljeu algorithm

A Simple general Case



Thus we see that for $t \in [1, 2]$ we have:

$$f(t) = p_1 \frac{(2-t)^2}{2} + p_2\left[\frac{(2-t)t}{2} + \frac{(2-t)(t-1)}{2}\right] + p_3(t-1)^2$$

Thus $f(t)$ is indeed a polynomial of degree 2.

Properties

From the evaluation procedure, certain properties are obvious:

- The point $f(t)$ is a convex combination of the control points. This is clear since in the modified de-Casteljeu/deBoor algorithm in every stage, new points are created which are convex combinations of earlier points, and so on.

- The second observation is locality. Note that if the evaluation is to be made at t and the relevant portion of the knot vector is:

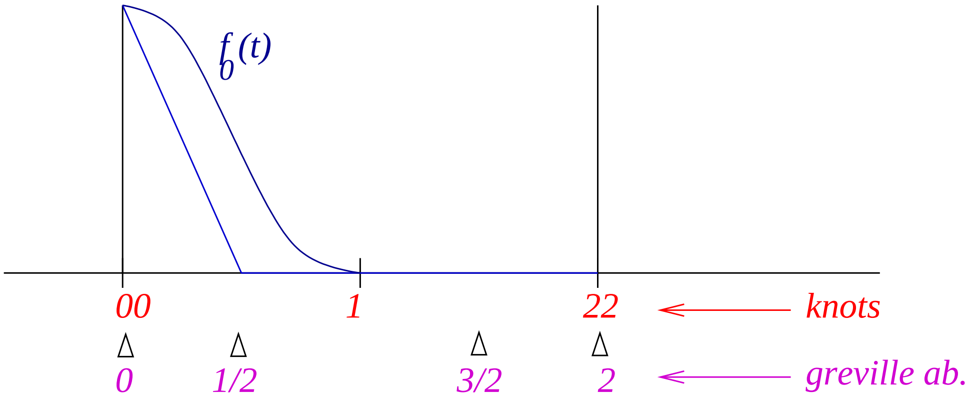
$$\ldots \leq \beta_{i-D+1} \leq \beta_{i_D+2} \leq \ldots \leq \beta_i \leq t \leq \beta_{i+1} \leq \ldots \leq \beta_{i+D}$$

  We then see that $\xi_{i-D+1}, \xi_{i-D+2}, \ldots, \xi_{i+1}$ are the only greville abscissas which will play a role. Whence $f(t)$ is completely determined by only a subset of the control points, viz. $\{p_{i-D+1}, \ldots, p_{i+1}\}$.

Basis Functions

What then are the basis functions?
So, let β be a knot vector, say $[00122]$, which we have seen needs 4 control points. The basis functions $f_i(t)$ for $i = 0, \ldots, 3$ correspond to the control polygons

$$\{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\}$$



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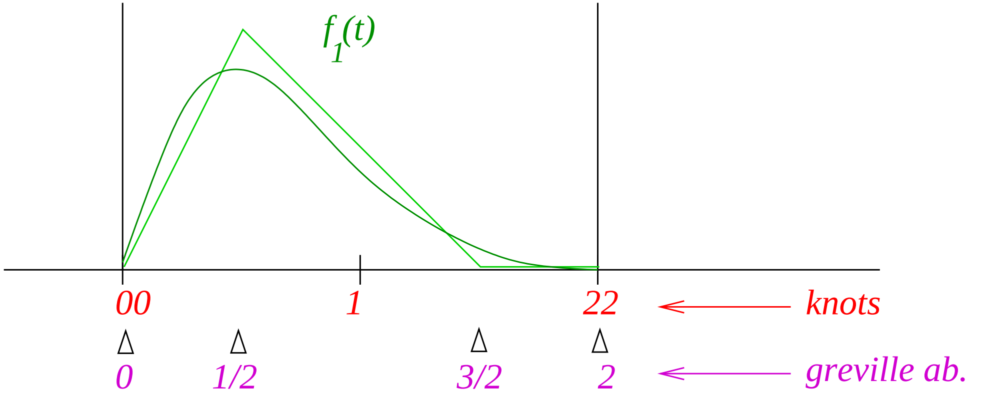
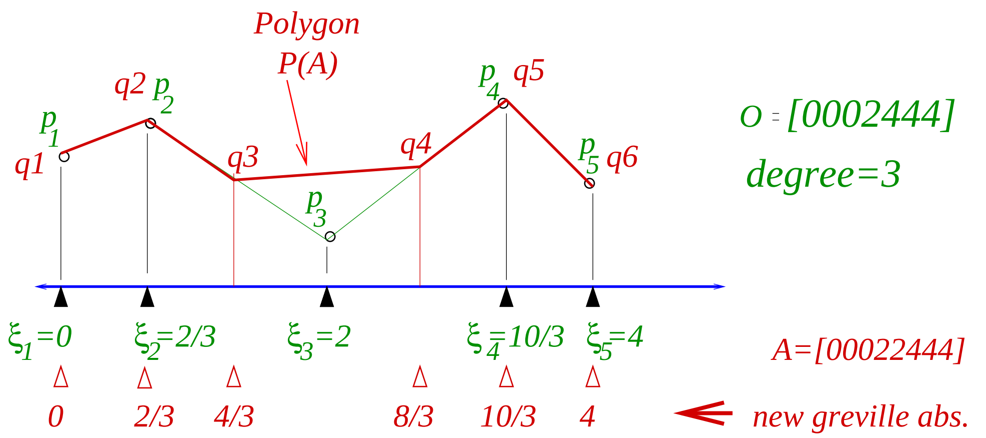
$$\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$$



Decoupling

Question: What is the connection between piece-wise Bezier and B-Spline? For the knot vector β, insert each β_i so that the multiplicity becomes D. Now read off the control points for each segment!



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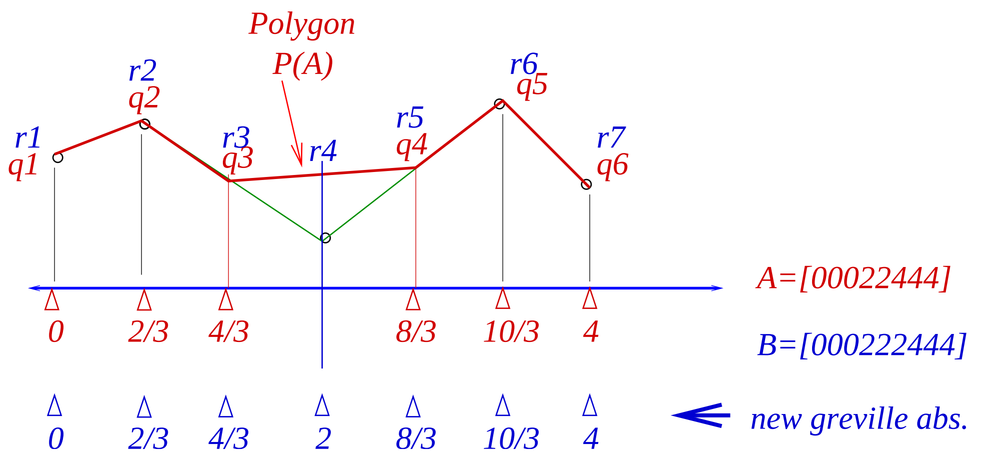

And again.

Decoupling

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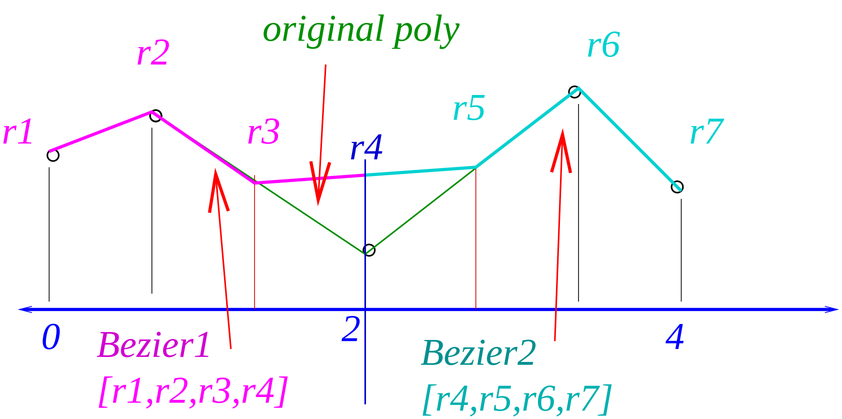

Finally, read off the control points.

Note the relationship between $[r_2, r_3, r_4]$ and $[r_4, r_5, r_6]$.

Wrap-Up

This covers our discussion of splines. See my notes for much of the mathematics behind it.
Things missing:

- End Conditions.
- Subdivision.
- Use in tensor-product surfaces.
- Plot of the basis functions.