Surfaces:Tensor Products

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Polynomials in 2 variables

Let $P^{m,n}[u,v]$ denote the vector space of all polynomials of degree atmost m in u and n in v. Thus, for example,

$$3u^2v - v^3 \in P^{2,3}[u,v] \subset P^{3,3}[u,v]$$

The dimension of $P^{m,n}[u,v]$ is obviously $(m+1)(n+1)$ and the Taylor basis for it is the set:

$$\{u^i v^j | 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

Just as polynomials in one variable served us to parametrize curves, these will serve us to parametrize surfaces.

Tensor-Product Bases

Actually, if $B = \{b_0(u), \ldots, b_m(u)\}$ is a basis for $P^m[u]$ and $C = \{c_0(v), \ldots, c_n(v)\}$ is a basis for $P^n[v]$ then:

$$B \otimes C = \{b_i(u)c_j(v) | 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

is a basis for $P^{m,n}[u, v]$.
Question :Show that elements of $B \otimes C$ are linearly independent.
Suppose that (as polynomials):

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij} b_i(u) c_j(v) = 0$$

Whence, for every u_0, we construct the polynomial:

$$p(u_0, v) = \sum_{j=0}^{n} (\sum_{i=0}^{m} \alpha_{ij} b_i(u_0)) c_j(v)$$

We see that $p(u_0, v) = 0$ for all v, whence *every* coefficient of $p(u_0, v)$ must be zero. In other words, for all j and u_0,

$$\sum_{i=0}^{m} \alpha_{ij} b_i(u_0) = 0$$

Since, b_i's are linearly independent, we are forced to conclude that $\alpha_{ij} = 0$ for all i and j.

□

In particular we have the: Bernstein Basis:

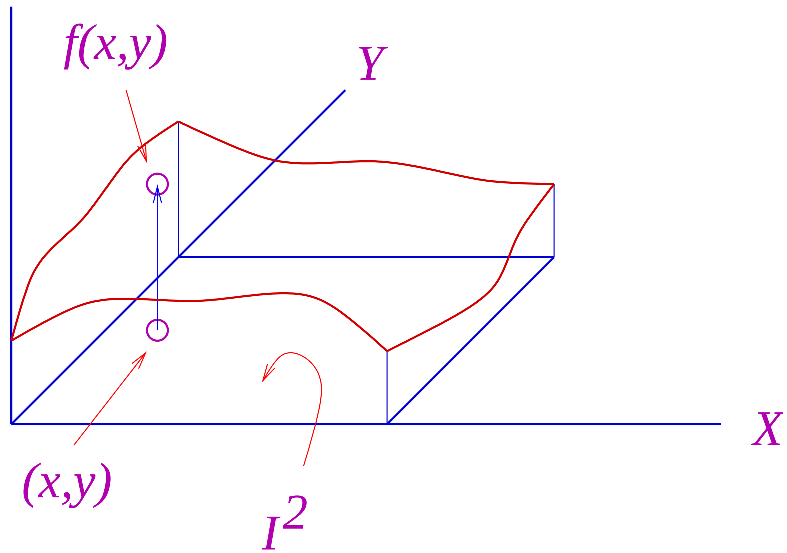
$$\{\binom{m}{i} u^i (1-u)^{m-i} \binom{n}{j} v^j (1-v)^{n-j} | 0 \leq i \leq m, \; 0 \leq j \leq n\}$$

We denote the typical basis element by $B_i^m(u) B_j^n(v)$.

Home Page
Title Page
Contents
Page 5 of 29
Go Back
Full Screen
Close
Quit

Functions and the Approximation Problem

I with denote the interval $[0,1]$ and I^2 the unit square $[0,1]\times[0,1]$. Let $f : I^2 \longrightarrow \mathbb{R}$ be a function on the unit square.



Is there a polynomial approximation to f?

The Bernstein-Weierstrass Approximation Theorem

Fix m and n, and form the data

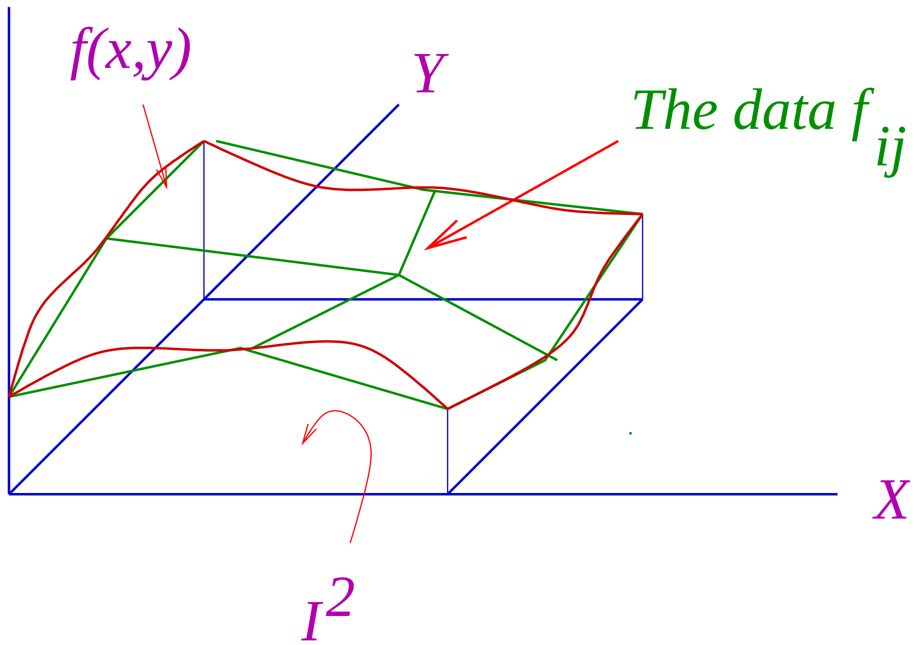
$$S = \{f_{ij} = f(\frac{i}{m}, \frac{j}{n}) | 0 \leq i \leq m,\ 0 \leq j \leq n\}$$

We define the Bernstein Approximation

$$B^{m,n}(f)(u,v) = \sum_i \sum_j f_{ij} B_i^m(u) B_j^n(v)$$

Theorem: Let f be a function on I^2, and let $\epsilon > 0$. Then there are m, n such that $|f(u,v) - B^{m,n}(f)(u,v)| < \epsilon$ for all $(u,v) \in I^2$.
Thus the 1-d situation has a complete 2-d analogue.

The Picture



Home Page

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 29

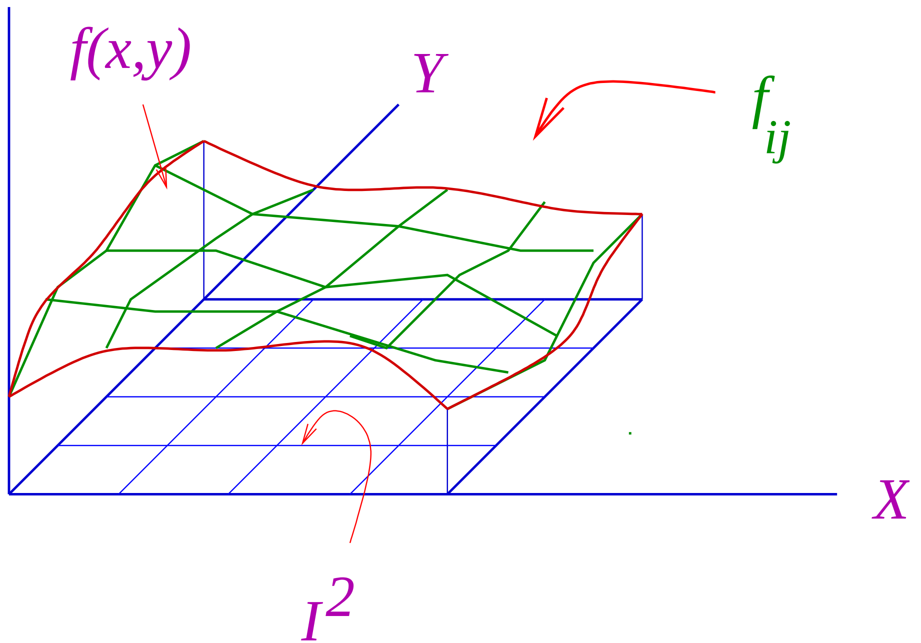
Go Back

Full Screen

Close

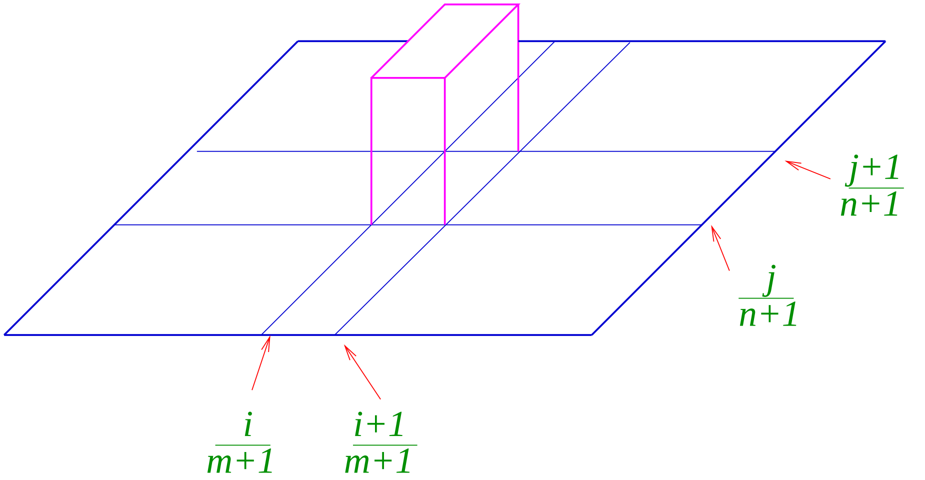
Quit

The Finer Picture



The Unit Step

As before it si convenient to associate $B_i^m(u)B_j^n(v)$ with the 2-dimensional unit step function below. The 'greville abscissa' is obviously $(\frac{i}{m}, \frac{j}{n})$ which occurs within the support of the step.



As expected $\int_0^1 \int_0^1 B_i^m(u)B_j^n(v)dudv = \frac{1}{(m+1)(n+1)}$.

The Control Polygon

We will now discard the function f.

Let S be an $m \times n$ matrix (in C++ notation, i.e., [0...m][0...n]) with entries in $\mathbb{R}$ (or $\mathbb{R}^3$).

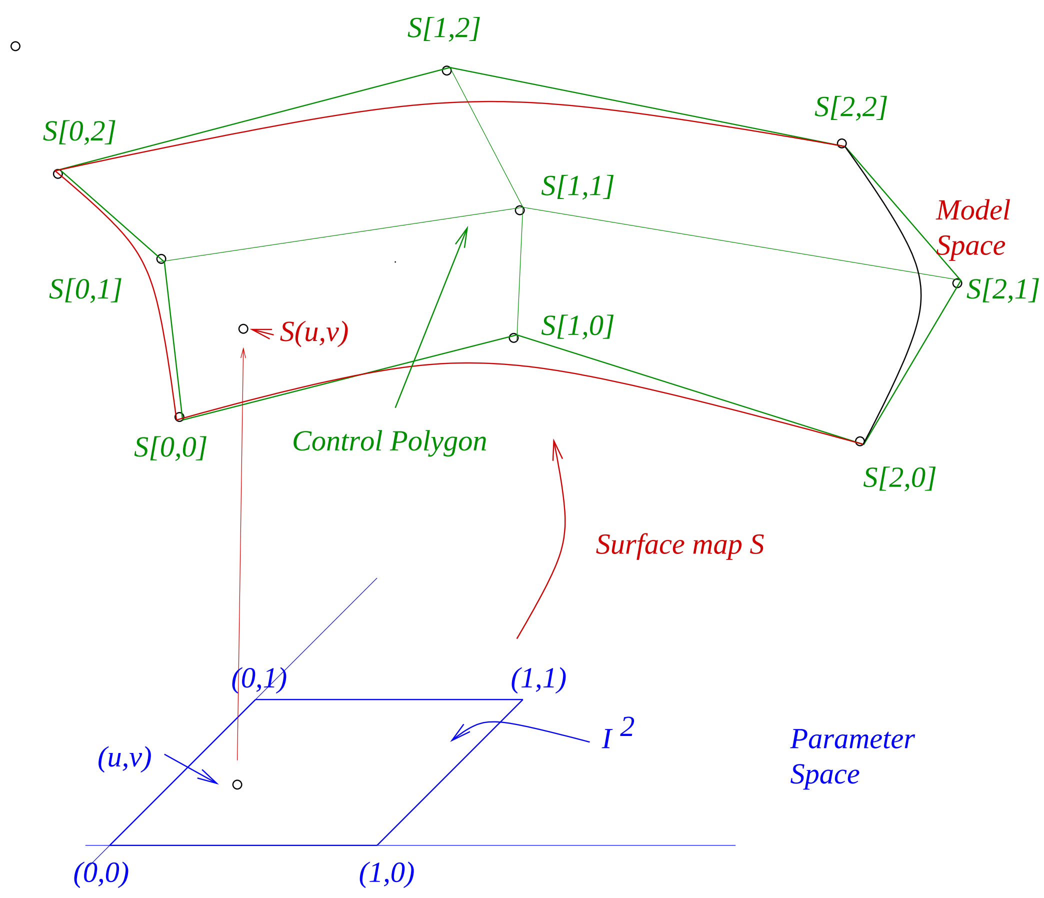
S is called the Control Polygon.

We define $S(u, v)$ as:

$$S(u, v) = \sum_i \sum_j S[i, j] B_i^m(u) B_j^n(v)$$

S will be called the tensor-product surface for the given control polygon.

An example


S[1,2]
S[0,2]
S[2,2]
S[1,1]
Model
Space
S[0,1]
S[2,1]
S(u,v)
S[1,0]
S[0,0]
Control Polygon
S[2,0]
Surface map S
(0,1)
(1,1)
I 2
Parameter
Space
(u,v)
(0,0)
(1,0)

Some Observations

$$\boxed{S(u,v) = \sum_i \sum_j S[i,j] B_i^m(u) B_j^n(v)}$$

Lets evaluate $S(0,0)$. Since $B_i^m(0) = 0$ unless $i = 0$ and $B_j^m(0) = 0$ unless $j = 0$, we have $S(0,0) = S[0,0]$. Similarly, we have the other 'corner points'. Thus:

$$\boxed{\begin{aligned} S(0,0) &= S[0,0] \\ S(1,0) &= S[m,0] \\ S(0,1) &= S[0,n] \\ S(1,1) &= S[m,n] \end{aligned}}$$

Boundary Curves

Next, lets look at $S(u, 0)$, which is the image of a boundary line of I^2. Again, since on this curve $v = 0$, we have $B^n_j(0) = 0$ for $j \neq 0$. Thus the sum reduces to:

$$S(u, 0) = \sum_{i=0}^{m} S[i, 0] B^m_i(u)$$

This is clearly the bezier curve corresponding to the first column of S as its control points.
In general, we have:

$$\begin{array}{|lcl|}\hline S(u,0) & = & \sum_{i=0}^{m} S[i,0]B^m_i(u) \\ S(u,1) & = & \sum_{i=0}^{m} S[i,n]B^m_i(u) \\ S(0,v) & = & \sum_{j=0}^{n} S[0,j]B^n_j(v) \\ S(1,v) & = & \sum_{j=0}^{n} S[m,j]B^n_j(v) \\ \hline \end{array}$$

Home Page

Title Page

Contents

Page 14 of 29

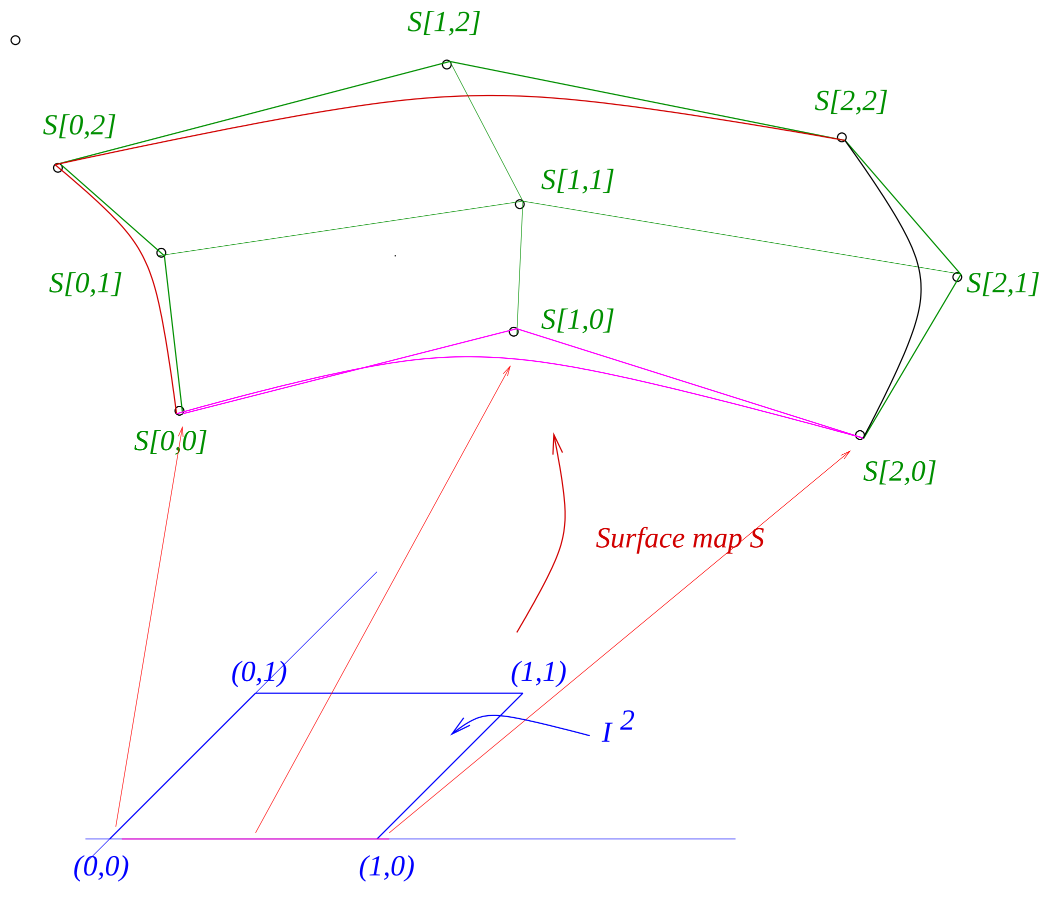
Go Back

Full Screen

Close

Quit

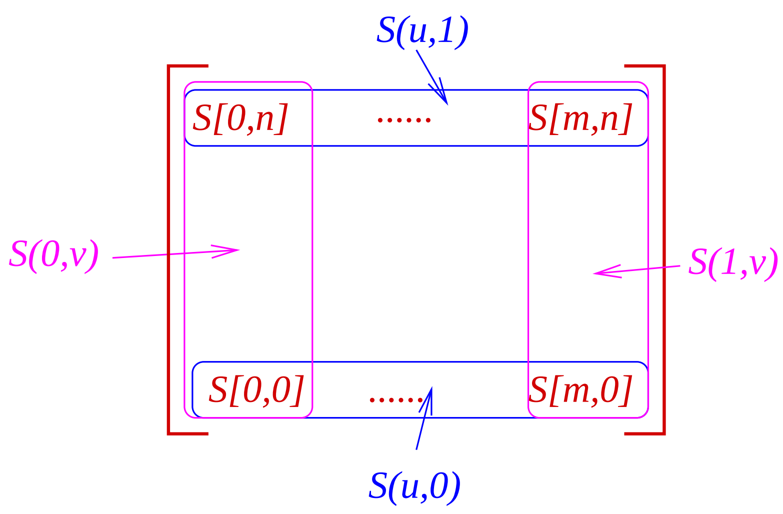
Pictorially



And Schematically

In terms of the control matrix, perhaps it is usefule to use the *french notation* and number rows and columns from the bottom left corner. Then, we have:

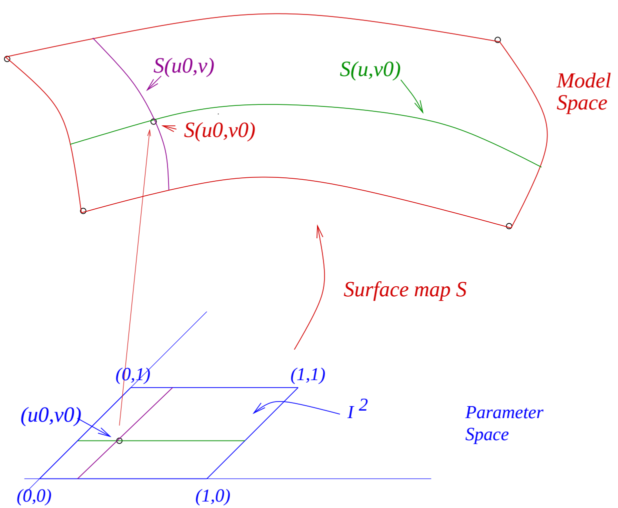
The Control Matrix S



Iso-parametric Lines

But what about general $S(u_0, v)$ for a fixed u_0 and $v \in [0, 1]$? Or $S[u, v_0]$ for a fixed v_0 but u ranging over $[0, 1]$?

These curves (in the model space) are called iso-parametric lines. Thus $S[u_0, v]$ is the iso-parametric line for $u = u_0$.



Iso-parametric Lines contd.

Lets evaluate $S(u_0, v)$. Re-arranging the sum $S(u, v)$, we see that:

$$S(u_0, v) = \sum_{j=0}^{n}[\sum_{i=0}^{m} S[i, j]B_i^m(u_0)]B_j^n(v)$$

We call $\sum_{i=0}^{m} S[i, j]B_i^m(u_0)$ as $S[u_0, j]$ and observe that $S(u_0, v)$ is a bezier curve with control points $[S[u_0, 0], S[u_0, 1], \ldots S[u_0, n]]$.

Also, note that *each* of these control points $S[u_0, j]$ is itself moving on a bezier curve parametrized by u.

Perhaps, the matrix notation is more convenient to observe this. We see that:

$$S(u,v) = [B_n^n(v), \ldots, B_0^n(v)] \begin{bmatrix} S[0,n] & \ldots & S[m,n] \\ \vdots & & \vdots \\ S[0,0] & \ldots & S[m,0] \end{bmatrix} \begin{bmatrix} B_0^m(u) \\ \vdots \\ B_m^m(u) \end{bmatrix}$$

This may be consicely written as $S(u,v) = B(v)SB(u)^T$. Consequently, forming the product as $S(u,v) = B(v)(SB(u)^T)$, we see that:

$$S(u_0,v) == [B_n^n(v), \ldots, B_0^n(v)] \begin{bmatrix} S[u_0,n] \\ \vdots \\ S[u_0,0] \end{bmatrix}$$

Also note that $\sum_i \sum_j B_i^m(u) B_j^n(v) = 1$ and thus $S(u,v)$ is a convex combination of the entries of S.

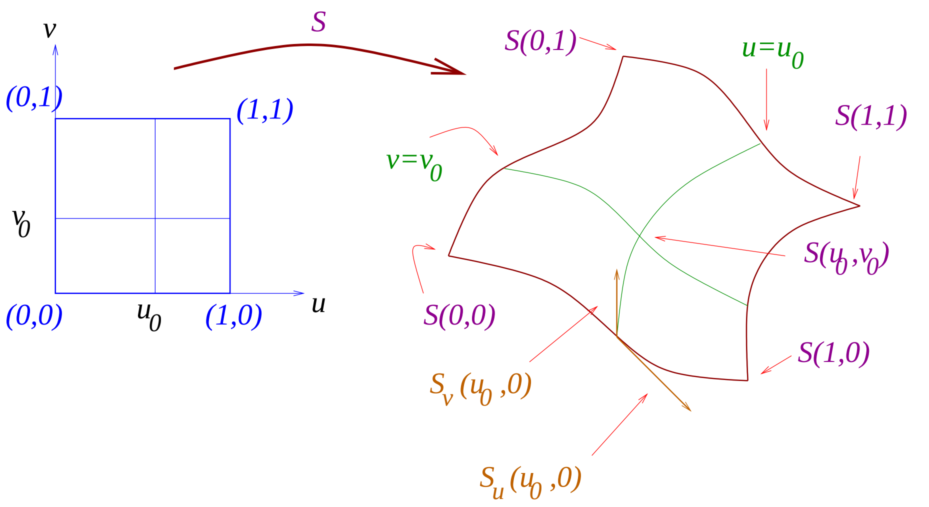
End Tangents and Normals

Given a map $S : I^2 \to \mathbb{R}^3$ as we have already determined the boundary $S(0, v), S(u, 0)$, and so on. Other important data is the first-order data, viz., the tangents.
For convenience, let us consider the boundary point $S(u_0, 0)$. At any boundary point, we have two tangents to compute.

$$\begin{aligned} S_u(u_0, 0) &= \lim_{u \to u_0} \frac{S(u,0) - S(u_0,0)}{u - u_0} \\ S_v(u_0, 0) &= \lim_{v \to u_0} \frac{S(u_0,v) - S(u_0,0)}{v} \end{aligned}$$

These two tangents are shown in the next picture.

An Example



The quantity $S_u(u_0, 0)$ is easily computed as the derivative of the boundary $S(u, 0) = \sum_{i=0}^{m} S[i, 0]B_i^m(u)$. We may thus use the curve-tangent law explained earlier to get:

$$S_u(u_0, 0) = m[\sum_{i=0}^{m-1}(S[i + 1, 0] - S[i, 0])B_i^{m-1}(u)]$$

$S_v(u_0, v)$

This quantity is a bit more delicate, since it is the tangent to the iso-parametric curve $S(u_0, v)$ at $v = 0$.
We have seen that:

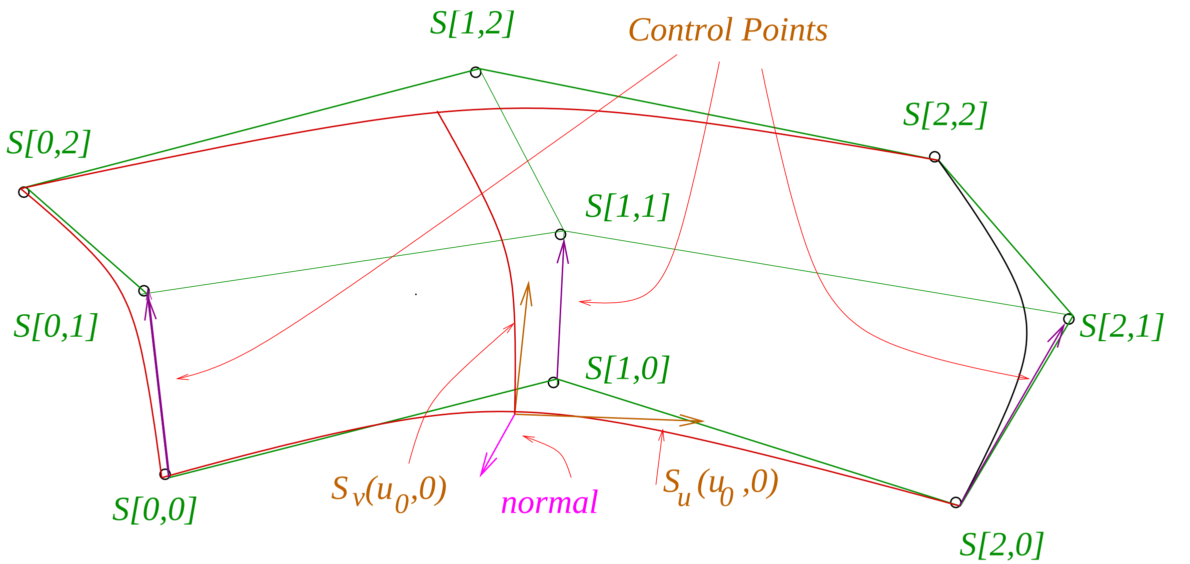
$$S(u_0, v) = \sum_{j=0}^{n} S[u_0, j] B_j^n(v)$$

where $S[u_0, j] = \sum_{i=0}^{n} S[i, j] B_i^m(u_0)$.
Thus $S_v(u_0, 0)$, the end-tangent to this curve, is $m(S[u_0, 1] - S[u_0, 0])$.
Back-substituting, we get:

$$\begin{aligned} S_v(u_0, 0) &= m[\sum_{i=0}^{n} S[i,1] B_i^m(u_0) - \sum_{i=0}^{n} S[i,0] B_i^m(u_0)] \\ &= m[\sum_{i=0}^{n} (S[i,1] - S[i,0]) B_i^m(u_0)] \end{aligned}$$

Thus $S_v(u_0, 0)$ is also a bezier with control points $[S[1,0] - S[0,0], S[1,1] - S[1,0], \ldots, S[m,1] - S[m,0]]$.

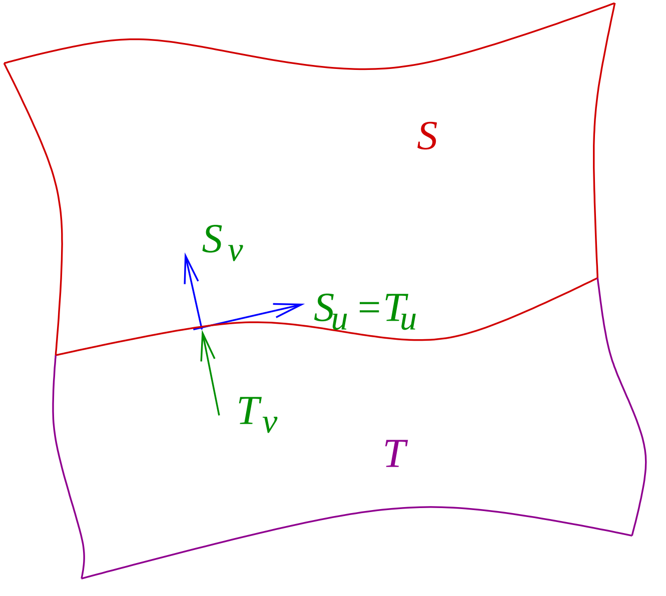
Pictorially



The normal at that point is given by the cross-product $S_v \times S_u$.

Splicing

Consider Two surfaces given by control points S and T. We would like to have them meet at a common boundary, and smoothly. Thus for example, we require $S(u,0) = T(u,1)$ for all $u \in [0,1]$. Furthermore, we require that the normals match too.



The conditions

The condition $S(u, 0) = T(u, 1)$ is easily satisfied by having the bottom row of S match the top row of T.

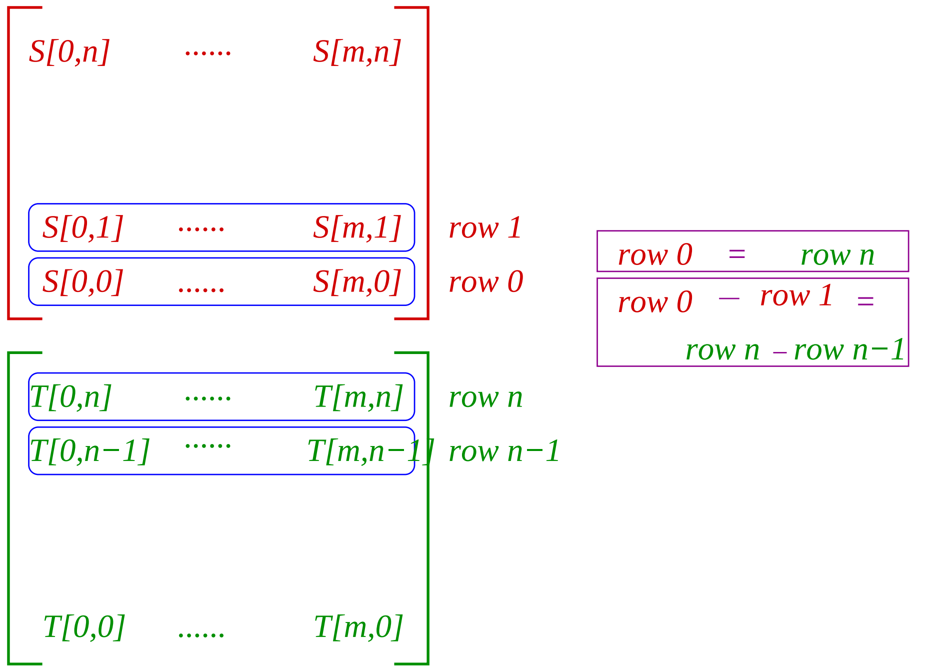
This will also ensure that $S_u = T_u$ since both are tangents to the same curve.

Lets examine the normal condition next. $S_u \times S_v \equiv T_u \times T_v$, is achieved if we force S_v to be a multiple of T_v. This is forced by fixing a multiple, say α and requiring that:

$$S[i, 1] - S[i, 0] = \alpha(T[i, n] - T[i, n - 1])$$

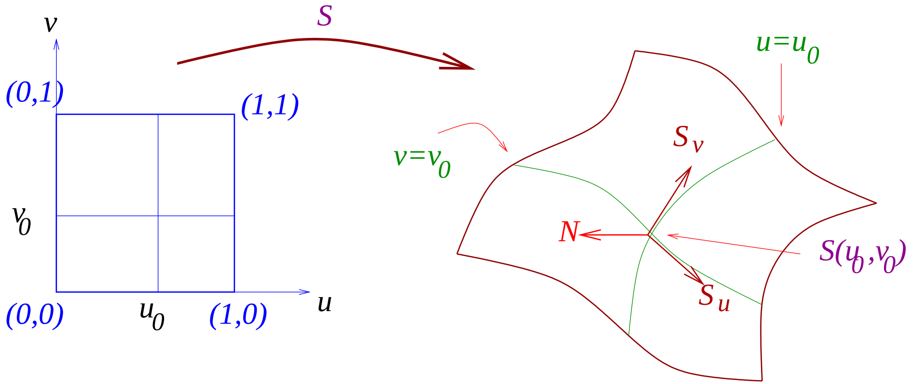
Home Page
Title Page
Contents
Page 25 of 29
Go Back
Full Screen
Close
Quit

Schematically



The Normal System

A surface, if part of a solid, has at *every* point, an outward normal. Thus, given a (u_0, v_0) we are now faced with specifying *uniformly* an outward normal at $S(u_0, v_0)$!.



Consider the figure above. At the point $S(u_0, v_0)$, we have the two tangents S_u and S_v. Let $N = S_u \times S_v$. Clearly the outward normal at $S(u_0, v_0)$ must be either N or $-N$.

The Sign of the Normal

We claim that if the outward normal at $S(u_0, v_0)$ is, say, $-N = -(S_u \times S_v)$, then it is so at every u, v [a].

Thus *all* that needs to be stored is a $sign \in \{+1, -1\}$. The normal at any point $S(u, v)$ is given by

$$sign \cdot (S_u \times S_v)$$

Proof: Let $U(u, v)$ be the unit outward normal which exists! Clearly, $U(u, v)$ is a smooth function on the surface.

Let $M(u, v) = sign \cdot \frac{S_u \times S_v}{|S_u \times S_v|}$. We see that (i) $M(u, v)$ is a smooth function on u, v, and (ii) $M(u, v)$ is normal at $S(u, v)$.

[a] provided $S_u \times S_v$ is never zero

Continued

Thus at all points (u, v), the vectors $M(u, v)$ and $U(u, v)$ are collinear. Now the proof goes in the following 3 steps:

- Since both are unit, we have $M(u, v)/U(u, v) \in \pm 1$.
- Since both U and M are smooth and unit, $M(u, v)/U(u, v)$ must be *uniformly* either $+1$ or -1.
- But we know that at (u_0, v_0) it is $+1$ and thus $M(u, v) = U(u, v)$.

$\square$

Things NOT covered

1. Surface Re-construction
2. Subdivision, Evaluation, Degree Elevation
3. Special Surfaces such as Coons-Patch
4. Tangent Planes, Gauss Map and Curvature