Geometric Complexity Theory Milind Sohoni¹

An approach to complexity theory via Geometric Invariant Theory.

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¹joint work with Ketan Mulmuley

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Talk Outline

- Complexity Theory Primer.
- Formula size and the permanent.
- The projective orbit closure
- Stability and partial stability.
- Obstructions.
- Two Questions.
- The subgroup restriction problem.

The satisfiability problem

- Boolean variables x_1, \ldots, x_n
- Term $t_1 = (\neg x_1 \lor x_3 \lor x_7)$, and so on upto t_m .
- Formula $t_1 \wedge t_2 \wedge \ldots \wedge t_m$

Question: Decide if there is a satisfying assignment to the formula.

There is no known algorithm which works in time polynomial in n and m.

Harder Question: Count the number of satisfying assignments. Thus we have the decision problem and its counting version.

Matchings

Question: Given a bipartite graph on *n*, *n* vertices, check if the graph has a complete matching.

This problem has a known polynomial time algorithm.



Harder Question: Count the number of complete matchings.

- There is no known polynomial time algorithm to compute this number.
- Even worse, there is no proof of its non-existence.

Thus, there are decision problems whose counting versions are hard.

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The permanent

If X is an $n \times n$ matrix, then the permanent function is:

$$perm_n(X) = \sum_{\sigma} \prod_i x_{i,\sigma(i)}$$

The relationship with the matching problem is obvious. When X is 0-1 matrix representing the bipartite graph, then perm(X) counts the number of matchings.

- There is no known polynomial time algorithm to compute the permanent, and worse, no proof of its non-existence.
- The function *perm_n* is *#P*-complete. In other words, it is the hardest counting problem whose decision version is easy to solve.

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Our Thesis

- Non-existence of algorithm structure (obstructions)
- These happen to arise in the GIT and Representation Theory of Orbits.

CAUTION: There are plenty of NP-complete problems in Representation Theory. But thats not what we are saying.

Example

• Hilbert Nullstellensatz : Either polynomials f_1, \ldots, f_n have a common zero, or there are g_1, \ldots, g_n such that

$$f_1g_1+\ldots+f_ng_n=1$$

• Thus g_1, \ldots, g_n obstruct f_1, \ldots, f_n from having a common zero.

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Other #P-complete problems

- Compute the Kostka number $K_{\lambda\mu}$.
- Compute the Littlewood-Richardson number $c_{\lambda,\mu}^{\nu}$.

Note that there are polynomial time algorithms to check non-zero-ness.

We will stick with the permanent.

- homogeneous polynomial, i.e., in $Sym^n(X)$,
- Distinguished stabilizer within *GL*(*X*):

perm(X) = perm(PXQ) $perm(X) = perm(D_1XD_2)$

• P, Q permutation matrices, D_1, D_2 diagonal matrices.

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Computation Model-Formula Size

Let $p(X_1, ..., X_n)$ be a polynomial. A formula is a particular way of writing it using * and +.

formula = formula*formula | formula+formula

- Thus the same function may have different ways of writing it.
- The number of operations required may be different.

Example:

•
$$a^3 - b^3 = (a - b)(a^2 + a * b + b^2).$$

• Van-der-Monde $(\lambda_1, \ldots, \lambda_n) = \prod_{i \neq j} (\lambda_i - \lambda_j).$

Formula size: the number of * and + operations.

• LHS1 is 5, RHS1 is 7, RHS2 is 2n.

Formula size

- A formula gives a formula tree.
- This tree yields an algorithm which takes time proportional to formula size.



Does $perm_n$ have a formula of size polynomially bounded in n? (This also implies a polynomial time algorithm)

Valiant's construction: converts the tree into a determinant.

Valiant's Construction

If $p(Y_1, \ldots, Y_k)$ has a formula of size m/2 then,

- There is an inductively constructed graph G_p with atmost m nodes, with edge-labels as (i) constants, or (ii) variable Y_i .
- The determinant $det(A_p)$ of the adjacency matrix of G_p equals p.



The general case.

A simple formula.

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Addition

The Matrix

In other words:

$$p(Y_1,\ldots,Y_k) = det_m(A)$$

where $A_{ij}(Y)$ is a degree-1 expression on Y. For our example, we have:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & y \\ 1 & 0 & 0 \end{bmatrix} det(A) = y$$

- Note that in Valiant's construction $A_{ij} = Y_r$ or $A_{ij} = c$.
- For general A_{ij} , since det_m is polynomial-time computable, we get an algorithm to compute p in time polynomial in m.

formula size = $m/2 \implies det_m(A) \implies$ algorithm polynomial in m

The homogenization

Lets homogenize the above construction:

- Add an extra variable Y_0 .
- Let $p^m(Y_0, \ldots, Y_k)$ be the degree-*m* homogenization of *p*.
- Homogenize the A_{ij} using Y_0 to A'_{ij} .

We then have: $p^m(Y_0, \ldots, Y_k) = det_m(A')$

For our small example:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & y \\ 1 & 0 & 0 \end{bmatrix} \quad A' = \begin{bmatrix} 0 & y_0 & 0 \\ 0 & y_0 & y \\ y_0 & 0 & 0 \end{bmatrix} \quad det(A') = y_0^2 y$$

The \leq_{hom}



• A may even be singular.





Program for f(Y)

If there is an efficient algorithm to compute g then we have such for f as well.

Suppose that $perm_n$ has a formula of size m/2. How is one to interpret Valiant's construction?

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The "inserted" permanent

Let X be the space of all $m \times m$ -matrices. For m > n, we construct a new function $perm_n^m \in Sym^m(X)$.



Thus $perm_n$ has been inserted into $Sym^m(X)$, of which $det_m(X)$ is a special element.

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Thus perm_n has been inserted into $Sym^m(X)$, of which $det_m(X)$ is a special element.

• formula of size m/2Conclusion implies $perm_n^m = det_m(A')$ $perm_n = det_m(A)$ • Use x_{mm} as the $perm_n^m \prec_{hom} det_m$ homogenizing variable

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Group Action and \leq_{hom}

Let $V = Sym^m(X)$. The group GL(X) acts on V as follows. For $T \in GL(X)$ and $g \in V$

$$g_T(X) = g(T^{-1}X)$$

Two notions:

- The orbit: $O(g) = \{g_T | T \in SL(X)\}.$
- The projective orbit closure $\Delta(g) = \overline{cone(O(g))}.$

If $f \leq_{hom} g$ then $f = g(A \cdot X)$, whence

- If A is full rank then f is in the GL(X)-orbit of g.
- If not, then A is approximated by elements of GL(X).

Thus, in either case,

 $f \preceq_{hom} g \implies f \in \Delta(g)$

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Valiant again

- Thus, we see that if $perm_n$ has a formula of size m/2 then $perm_n^m \in \Delta(det_m)$.
- On the other hand, perm^m_n ∈ Δ(det_m) implies that for every ε > 0, there is a T ∈ GL(X) such that ||(det_m)_T - perm^m_n|| < ε. This yields a poly-time approximation algorithm for the permanent

Thus, we have an almost faithful algebraization of the formula size construction.

To show that $perm_5$ has no formula of size 10, it suffices to show:

$$perm_5^{20}
ot\in \Delta(det_{20})$$

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The Obstruction

So let $g, f \in V = Sym^d(X)$. How do we show that $f \notin \Delta(g)$.

Exhibit a homogeneous polynomial μ ∈ Sym^r(V*) which vanishes on Δ(g) but not on f.

This μ is then the required **obstruction**. We would need to show that:

•
$$\mu(f) \neq 0$$
.

•
$$\mu(g_T) = 0$$
 for all $T \in SL(X)$.

Check μ on every point of Orbit(g)

False start: Use the SL(X)-invariant elements of $Sym^d(V^*)$ for constructing such a μ .

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Invariants

- V is a space with a group G acting on V.
- $Orbit(v) = \{g.v | G \in G\}.$
- Invariant is a function $f: V \to \mathbb{C}$ which is constant on orbits.

Existence and constructions of invariants has been an enduring interest for over 150 years.

Example:

- V is the space of all $m \times m$ -matrices.
- $G = GL_m$ and $g.v = gvg^{-1}$.
- Invariants are the coefficients of the characteristic polynomial.

Invariants and orbit separation

To show that $f \notin \Delta(g)$

Exhibit a homogeneous *invariant* μ which vanishes on g but not on f. This μ would then be the desired obstruction.

- Easy to check if a form is an invariant.
- Easy to construct using age-old recipes.
- Easy to check that $\mu(g)=0$ and $\mu(f)
 eq 0.$

$$\mu(g) = 0 \implies \mu(g_T) = 0 \implies \mu(\Delta(g)) = 0$$

Important Fact

If g and f are stable and $f \notin \Delta(g)$, then there is a homogeneous invariant μ such that $\mu(g) \neq \mu(f)$.

What is Stability

- g is stable iff SL(X)-Orbit(g) is Zariski-closed in V.
- Most polynomials are stable.
- It is difficult to show that a particular form is stable.
- **Hilbert** : Classification of unstable points.

$perm_m$ and det_n are stable.

Proof: Kempf's criteria: If $v \in V$ is unstable then there is a one-parameter subgroup $\lambda(t)$ of SL(X) such that:

- $\lim_{t\to 0} \lambda(t) \cdot v$ exists and is outside Orbit(v).
- λ(t) commutes with the the stabilizer H of v within SL(X).

The limit is exhibited by a 1-parameter subgroup commuting with *H*.

Rich Stabilizers

The stabilizer of the determinant:

- The form $det_m(X)$:
 - $\begin{array}{c} \bullet \quad X \to AXB \\ \bullet \quad X \to X^T \end{array}$
- det_m ∈ Sym^m(X) determined by its stabilizer.

The stabilizer of the permanent:

- The form $perm_m(X)$:
 - $X \rightarrow PXQ$
 - $\blacktriangleright X \to D_1 X D_2$
- perm_m ∈ Sym^m(X)
 determined by its stabilizer.

The huge stabilizers and the uniqueness of the form in $Sym^m(X)$ along with Kempf suffice to show the stability of det_m and $perm_m$.

Tempting to conclude that the homogeneous obstructing invariant μ now exists.

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The Main Problem

Recall we wish to show

 $perm_n^m \not\in \Delta(det_m)$

where

 $perm_n^m = x_{mm}^{m-n} perm(X').$

 $perm_n^m$ is unstable, in fact in the null-cone, for very trivial reasons.

- Added an extra degree equalizing variable.
- Treated as a polynomial in a larger redundant set of variables.

Here is the 1-*PS*, with suitable a, b > 0: • usefull $\times t^{a}$ • useless $\times t^{-b}$ whence

 $\lim_{t\to 0}\lambda(t)\cdot perm_n^m\to 0_V$



Obstructions

- Thus every invariant μ will vanish on $perm_n^m$.
- There is no invariant μ such that $\mu(det_m) = 0$ and $\mu(perm_n^m) \neq 0$.

Homogeneous invariants will never serve as obstructions.

Two Questions:

- Is there any other system of functions which vanish on $\Delta(det_m)$?
- Can anything be retrieved from the superficial instability of perm^m_n?

Question 1

Is there any other system of functions which vanish on $\Delta(det_m)$?

- We use the stabilizer $H \subseteq SL(X)$ of det_m .
- For a representation V_λ of SL(X), we say that V_λ is H-admissible iff V^{*}_λ|_H contains the trivial representation 1_H.

For g stable:

Fact:
$$k[Orbit(g)] \cong k[G/H] \cong \sum_{\lambda} H_{\text{-admissible}} n_{\lambda} V_{\lambda}$$

Observation: $k[\Delta(g)] \cong \sum_{\lambda \text{ H-admissible}} m_{\lambda} V_{\lambda}$

Thus a fairly restricted class of G-modules will appear in $k[\Delta(g)]$. We use this to generate some elements of the ideal for $\Delta(g)$.

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G and H

Consider next the *G*-equivariant surjection:

$$\phi: k[V] \to k[\Delta(g)]$$

We see that (i) ϕ is a graded surjection, and (ii) if $V_{\mu} \subseteq k[V]^d$ is not *H*-admissible, then $V_{\mu} \in ker(\phi)$. Let Σ_H be the ideal generated by such V_{μ} within k[V].

How good is Σ_H ?

G-separability: We say that $H \subseteq G$ is G-separable, if for every non-trivial H-module W_{α} such that:

• W_{α} appears in some restriction $V_{\lambda}|_{H}$.

then there exists a *H*-non-admissible V_{μ} such that $V_{\mu}|_{H}$ contains W_{α} .

The Local Picture

Theorem: Let g and H be as above, with (i) g stable, (ii) g only vector in V with stabilizer H, and (iii) H is G-separable. Then for an open subset U of V, $U \cap \Delta(g)$ matches $(k[V]/\Sigma_H)_U$.

Proof:

- Use the Luna-model $G \times_H X_N$ in a *G*-invariant open neighborhood of *Orbit*(*g*).
- Use G-separbility to isolate the "thick-zero-section" in Y.
- Use *Cartan-Oka* to lift this to $U \subseteq V$.

Applying this ...

The conditions: (i) stability of g, (ii) $V^H = \langle g \rangle$ and (iii) *G*-separability of *H*.

- *det_m* and *perm_n* satisfy conditions (i) and (ii) above.
- For n = 2, stabilizer of det_2 is indeed SL_4 -separable.
- For $V = \bigwedge^d$ and g the highest weight vector, $\Delta(g)$ is the grassmanian. For this Σ_H generates the ideal.
- For $g = det_m$, the data Σ_H does indeed *enter* the null-cone.

Still open:

- Look at $H = SL_n \times SL_n$ sitting inside $G = SL_{n^2}$. Is H G-separable?
- Does Σ_H determine $\Delta(det_m)$?

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Question 1 again

Is there any other system of functions which vanish on $\Delta(det_m)$?

- Stabilizer yields a rich set Σ_H of relations vanishing on $\Delta(det_m)$.
- Given G-separability, Σ_H does determine $\Delta(det_m)$ locally.

If $perm_n^m \in \Delta(det_m)$ then:

• Look at the injection $\Delta(perm_n^m) \rightarrow \Delta(det_m)$.

 $V_{\mu} \subseteq k[\Delta(perm_n^m)]$ and V_{μ} non-*H*-admissible, then V_{μ} is the required obstruction.

If $k[\Delta(perm_n^m)]$ is understood then this sets up the representation-theoretic obstruction.

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Question 2

Can anything be retrieved from the superficial instability of $perm_n^m$?

- Let's consider the simpler function $f = perm(X') \in Sym^n(X)$, i.e., with useful variables X' and useless X - X'.
- Let parabolic $P \subseteq GL(X)$ fix X'.
- P = LU, with U the unipotent radical.



We see that:

- f is fixed by U.
- f is L-stable.

Partially Stable

Definition: Let V be a G-module. Vector $v \in V$ is called partially stable if there is a parabolic P = LU and a regular $R \subseteq L$ such that:

- v is fixed by U, and
- v is R-stable.

For the form $perm_m^n(X) = y^{m-n}perm_n(X')$,

- P is that fixing (X', y) where y is the homogenizing variable.
- R is $GL(X') \times GL(y) \times GL(X X')$.

The form f

Recall $f = perm(X') \in V = Sym^n(X)$ and P fixing X'. We see that f is partially stable with $R = L = GL(X') \times GL(X - X')$.

With $W = Sym^n(X')$, we have the *P*-equivariant diagrams:

where $\Delta_W(f)$ is the projective closure of the GL(X')-orbit of f, and $\Delta_V(f)$ is that of the GL(X)-orbit of f.

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Through the parabolic

Since U acts trivially on W and $\Delta_W(f)$, we have the P-equivariant maps:

$$egin{array}{ccc} (k[\Delta_V(f)]^d) & \stackrel{\iota^*}{ o} & k[\Delta_w(f)]^d \ (k[\Delta_V(f)]^d)^{*U} & \stackrel{\iota}{\leftarrow} & k[\Delta_w(f)]^d \end{array}$$

In fact, for a GL(X)-module $Z^* \subseteq k[\Delta_V(f)]^d$, we see that

- $\iota^*(Z^*) \subseteq k[\Delta_W(f)]^d$ is non-zero.
- Z^U if non-zero, is the *bottom-most L*-module in $Z|_P$.
- Such modules Z^U equal the image of ι .

Stable versus partially-stable

- The GL(X)-module V^{*}_μ(X) occurs in k[Δ_V(f)]^d iff (V_μ(X))^U is non-zero. Thus V_μ(X') must exist.
- Next, the multiplicity of V_μ(X) in k[Δ_V(f)]^{d*} equals that of V_μ(X') in k[Δ_W(f)]^{d*}].

Now recall that $f = perm_m(X')$, and let $K = stabilizer(f) \subseteq GL(X')$.

But f is GL(X')-stable, and

 the GL(X')-modules which appear in k[Δ_W(f)]^d must be K-admissible.

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The Grassmanian

Consider $V = V_{1^k}(\mathbb{C}^m) = \bigwedge^k(\mathbb{C}^m)$ and the highest weight vector v.

- v is stable for the $GL_k \times GL_{m-k}$ action.
- $\Delta_V(v)$ is just the grassmanian.
- v is partially stable with the obvious P.
- $W = \mathbb{C}^k \subseteq \mathbb{C}^m$ and $\Delta_W(v)$ is the line through v.

whence

$$k[\Delta_W(v)] = \sum_d \mathbb{C}$$

• The above theorem subsumes the Borel-Weil theorem:

$$k[\Delta_V(v)]\cong \sum_d V_{d^k}(\mathbb{C}^m)$$

The partially stable case

Recall: Let V be a G-module. Vector $v \in V$ is called partially stable if there is a parabolic P = LU and a regular $R \subseteq L$ such that:

- v is fixed by U, and
- v is R-stable.

In the general case, there is a *regular* subgroup $R \subseteq L$, whence the theory goes through

$$\Delta_W(v) o \Delta_Y(v) o \Delta_V(v)$$

- The first injection goes through a Pieri branching rule.
- The second injection follows the lifting theorem.

In Summary

The General Conclusion

In other words, the theory of partially-stable $\Delta_V(f)$ lifts from that of the stable case $\Delta_W(f)$.

The crucial problem therefore is to understand $\Delta_W(f)$ or $\Delta_V(g)$, i.e., the stable case. There, for its geometry, we have:

- Is the stabilizer H of g, G-separable?
 - Larsen-Pink: do multiplicities determine subgroups?
- Does Σ_H generate the ideal of $\Delta(g)$?

The Representation-theoretic Obstruction

Let $H \subseteq GL(X)$ stabilize det_m and $K \subseteq GL(X')$ stabilize $perm_m^n$.

The representation-theoretic obstruction $V^*_{\mu}(X)$ for $perm^m_n \in \Delta(det_m)$

- $V_{\mu}(X)|_{H}$ must have a *H*-fixed point.
- $V_{\mu}(X)^U$ is non-zero.
- $V_{\mu}(X', y)|_R$ does not have a K-fixed point.

The subgroup restriction problem

- Given a G-module V, does $V|_H$ contain 1_H ?
- Given an *H*-module *W*, does $V|_H$ contain *W*?

The Kronecker Product Consider $H = SL_r \times SL_s \rightarrow SL_{rs} = G$, when does $V_{\mu}(G)$ contain an *H*-invariant?

This, we know, is a very very hard problem. But this is what arises (with r = s = m) when we consider det_m .

Any more geometry?

Is there any more geometry which will help?

- The Hilbert-Mumford-Kempf flags: limits for affine closures.
 - Extendable to projective closures?
 - Something there, but convexity of the optmization problem breaks down.
- The Luna-Vust theory: local models for stable points.
 - Extendable for partially stable points?
 - A finite limited local model exists, but no stabilizer condition seems to pop out.

Specific to Permanent-Determinant

Negative Results

- von zur Gathen: $m > c \cdot n$
 - Used the singular loci of *det* and *perm*.
 - Combinatorial arguments.
- Raz: multilinear model, but m > poly(n).
- Ressayre-Mignon: $m > c \cdot n^2$
 - Used the curvature tensor.

Positive results

• Jerrum-Sinclair-Vigoda: The permanent can be approximated by a randomized algorithm

The Subgroup Restriction Problem

The Kronecker Product Consider $H = SL_r \times SL_s \rightarrow SL_{rs} = G$, when does $V_{\mu}(G)$ contain an *H*-invariant?

Similar(?) problems recently solved (in Littlewood-Richardson coefficients $c_{\lambda,\mu}^{\nu}$):

- The *PRV*-conjecture on the non-zeroness of certain $c_{\lambda,\mu}^{\nu}$.
 - The Drinfeld-Jimbo quantized Lie algebras.
 - The crystal bases for modules.
- The saturation conjecture: $c_{n\lambda,n\mu}^{n\nu} \neq 0 \implies c_{\lambda,\mu}^{\nu} \neq 0$.
 - A polynomial-time non-combinatorial algorithm to detect if $c^{\nu}_{\lambda,\mu} \neq 0.$

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The Subgroup Quantization

The Kronecker Product For $X = V \otimes W$, is there a quantized structure and an injection $H = GL_q(V) \times GL_q(W) \rightarrow GL_q(X) = G$

- For the standard quantizations, no such injection exists.
- However, there maybe a different quantization $GL_q(\overline{X})$ which:
 - for which the injection above is natural
 - is (co)-semisimple
 - which has a representation theory following $GL_q(X)$

This is investigated in GCT 4 (under preparation).

In Conclusion

• Complexity Theory questions and projective orbit closures.

- stable and partially stable points.
- obstructions
- obstruction existence
 - Distinctive stabilizers
 - local definability of Orbit(g)
- partial stability
 - lifting theorems
- subgroup restriction problem
 - tests for non-zero-ness of group-theoretic data

Thank you.

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