CS 435, Linear Optimization

Tutorial 2, 2016

1. Let $x_1, \ldots, x_m \in \mathbb{R}^n$ with $m > n$. Show that there are $\alpha_i \in \mathbb{R}$, not all zero, such that $\sum_{i=1}^m \alpha_i x_i = 0$. In other words, m vectors in $\mathbb{R}^n$ where $m > n$ are linearly dependent. Use induction.

2. Use the above to show that if $x_1, \ldots, x_m \in \mathbb{R}^n$ with $m > n + 1$ then there are $\lambda_i \in \mathbb{R}$ such that $\sum_{i=1}^m \lambda_i = 0$ and $\sum_{i=1}^m \lambda_i x_i = 0$.

3. Use this to show that if $x \in conv(S)$, where $S \subseteq \mathbb{R}^n$ then there are points $s_1, \ldots s_{n+1}$ such that x is a convex combination of these points.

4. Let $p, x, y \in \mathbb{R}^n$ and let d be the distance function. Let $(p-x)\cdot(y-x) > 0$. Then show that there is a point $y' \in [x, y]$ such that $d(p, y') < d(p, x)$. Also, show that this y' is constructible.

5. If P is a finite set of points in $\mathbb{R}^n$ and $x \in conv(P)$ and $p \in \mathbb{R}^n$, then show that either x is the closest point to p in $conv(P)$ or there is a $q \in P$ such that $(p - x) \cdot x^T < (p - x) \cdot q^T$.

6. Examine `closest.sci`. Prove that the sequence so generated converges to the closest point.

7. Let P be the octahedron in 3-dimension and let $x = (0.1, 0.2, 0.3)$ and $p = (0.4, 0.4, 0.4)$. Find the next two iterations.

8. Experiment with `closest.sci` with various P and p. Try the octahedon with $p = (0.6, 0.6, 0)$.

9. Modify the algorithm so that it maintains a convex combination for the points x_n. Is there a way to guess the limit point?

10. Let D be given by the following system:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \\ 3 \\ 0 \\ 5 \\ 5 \\ 5 \\ 6 \end{bmatrix}$$

Consider the points $x_1 = (0,0,0), x_2 = (3,2,1), x_3 = (2.5,2.5,1)$ and $x_4 = (2,2,2)$ and directions $v_1 = (1,0,0), v_2 = (1,1,1), v_3 = (1,-1,0), v_4 = (-1,1,0)$. For each tuple (x_i, v_j) examine if v_j is a valid direction for x_i and if so, the maximal λ so that $x_i + \lambda v_j$ remains feasible.

11. Recall the octahedron given by the convex hull of the points $\{(0,0,\pm 1), (0,\pm 1,0), (\pm 1,0,0)\}$. Let $x_1 = (1,0,0)$ and $x_2 = (0.5,0.5,0)$. Compute $Dir(x_i)$, i.e., give an outline of how you would determine if $v \in Dir(x_i)$. Similarly compute $Normal(x_i)$.

12. Let $S \subseteq \mathbb{R}^n$. Let *cone*(S) be the collection of all conical combinations of the elements of S. Show *cone*(S) is a cone. Show that it is the smallest cone containing S.

13. Let C_1, C_2 be cones and let $C = cone(C_1, C_2)$. If C_1', C_2' are the polars of C_1, C_2 resp., express the polar of C in terms of the polars of C_1, C_2.

14. For the polytope D above and the points x_i, determine the objective directions c which will be optimal at the given points. In other words, determine $Normal(x_i)$. For the objective vector $c = (1,2,3)$ determine if $c \in Normal(x_i)$ and if not, find a favourable direction of motion $v \in Dir(x_i)$.