03/03/2014

Recall: • Defn of Convex set containing S $(S \subseteq \mathbb{R}^n)$

$$= \text{conv}(S) = \{ z \mid z = \sum \lambda_i x_i \text{ s.t. } \lambda_i \in S, \ \lambda_i \geq 0, \ \sum \lambda_i = 1 \}$$

• Defn of a half space

• Recall (Tutorial problem): $\text{conv}(S) = \bigcap_{\mathcal{H} \supseteq S} \mathcal{H}$

half spaces containing S.

• Defn of a Cone.

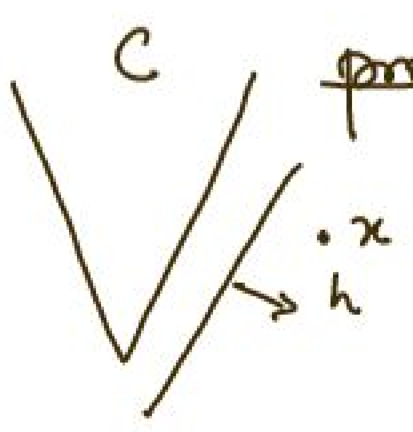
• Pigment Q'n

• Multicut knapsack problem

• Recall also $C \subseteq \mathbb{R}^n$ closed convex & $x \notin C$ then by H.B theorem then $\exists h \in \mathbb{R}^n$, $b \in \mathbb{R}$ $\quad h(x) > b \geq h(y) \ \forall y \in C$

Today:

Claim: If C is a cone then wlog $b = 0$



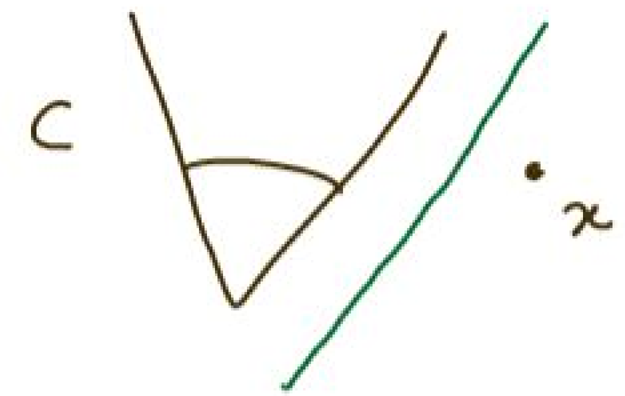
proof: Clear that $b \geq h(y)$ $\forall y$ and since $\overline{0} \in C$ then $b \geq 0$.

Suppose $\exists y \in C$ s.t. $h(y) > 0$

but $h(y) \le b$. But then $\exists \lambda > 0$ s.t.

$$h(\lambda y) = \lambda \cdot h(y) > b$$

Given a cone and a point outside it, say x



Suppose you give a plane not passing through the origin - say given by the green hyperplane.

The separator can be moved all the way till it passes through the origin & still being a separator.

H-B (Cone): If C is closed convex cone $x \notin C$ then there exists h s.t. $h(x) > 0$, $h(y) \le 0$ $\forall y \in C$.

$f_i(x_0) \le b_i \qquad i = 1, \ldots, k$

x_0 is a feasible point. $I(x_0) = \{1, 2, \ldots, r\}$

min $\quad c(x) \qquad$ (convex)

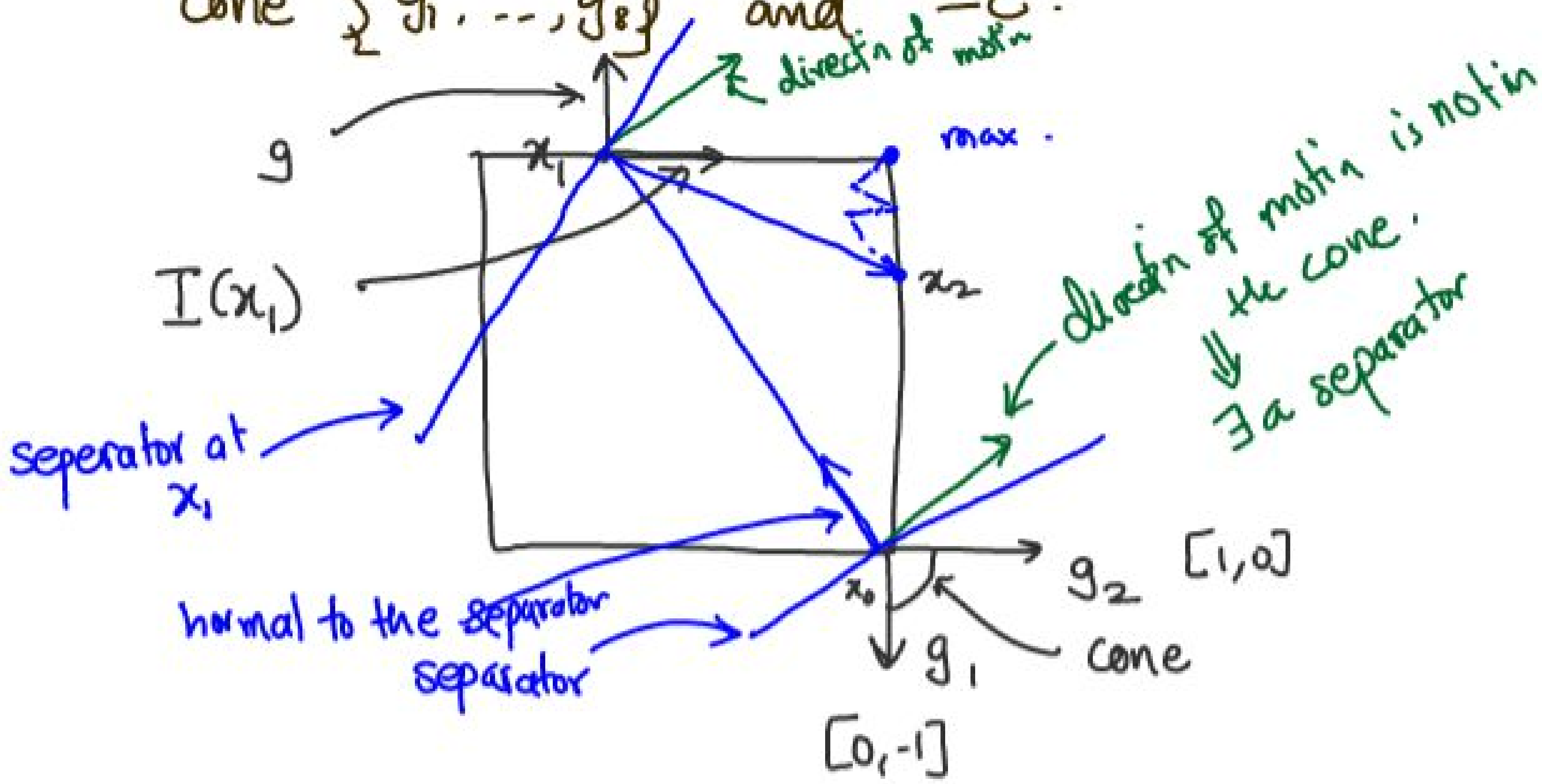
max $\quad -c(x) \qquad$ (concave)

$g_i = \nabla f_i(x_0)$

$-c = \nabla c(x_0)$

<u>Qn.</u>: Is there an optimizing direct'n : $\exists$ a y : $-c(y) > 0$, $g_i(y) \le 0$

- the feasible direction is a separator between cone $\{g_1, \dots, g_s\}$ and $-c$.



<u>KKT</u> : $g_1, \dots, g_r$ as before then

(i) either $-c \in \text{cone}(\{g_1, \dots, g_r\})$

or

(ii) $\exists$ a v : direction of motion such that

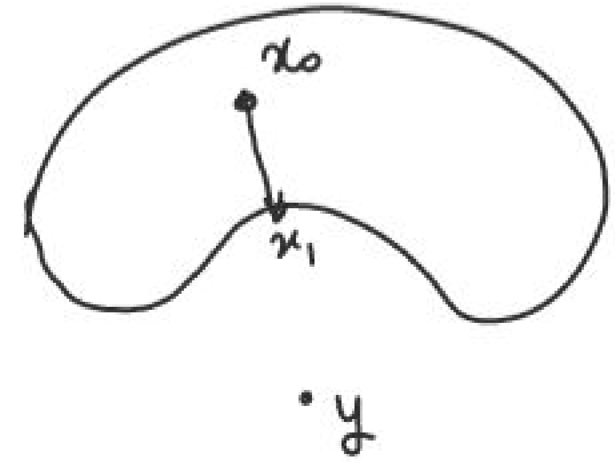
$$c(x + \epsilon v) < c(x)$$

Further $v \cdot -c > 0$, $v \cdot g_i \le 0 \;\; \forall i$

Thus, v is a separator between

$$-\text{grad}(c), \{\text{grad}(f_1), \dots, \text{grad}(f_r)\}$$

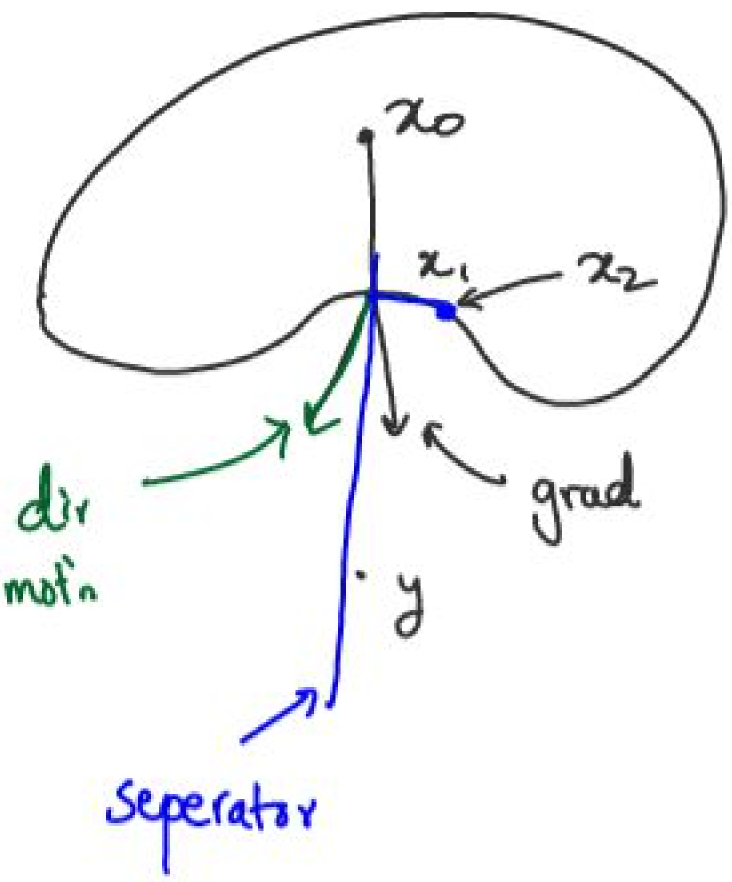
<u>Example</u>.



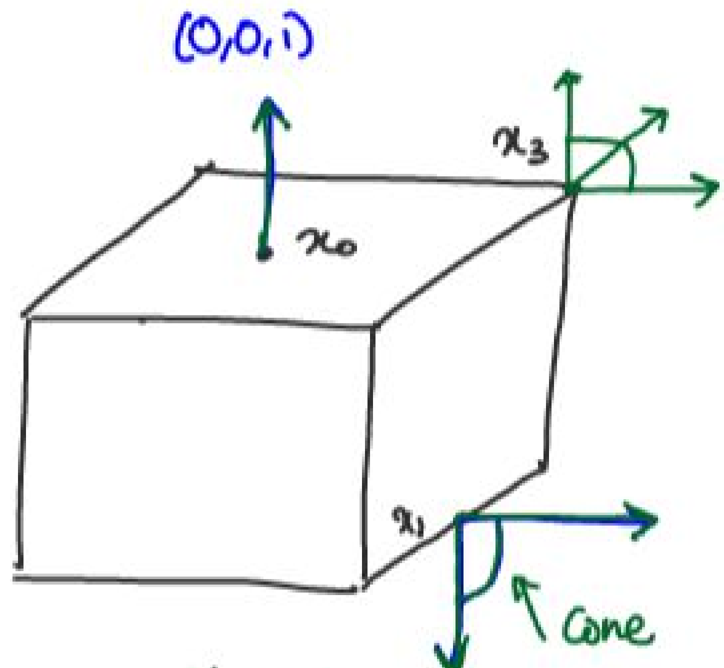
<u>Objective</u>: Minimize distance to y

$I(x_0) = \phi$

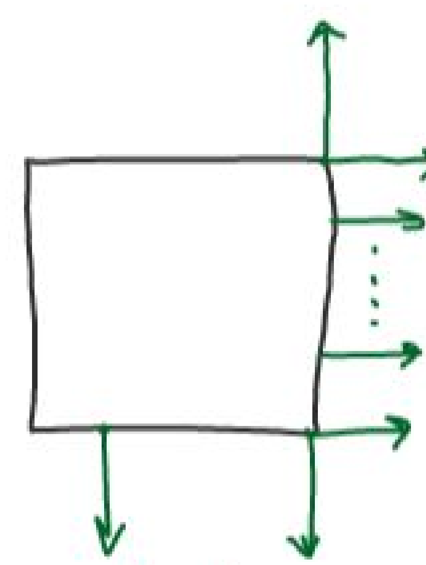
proceed like this till direct'n of mot'n coincides with the grad.



Example



Normals in the case of cubes.

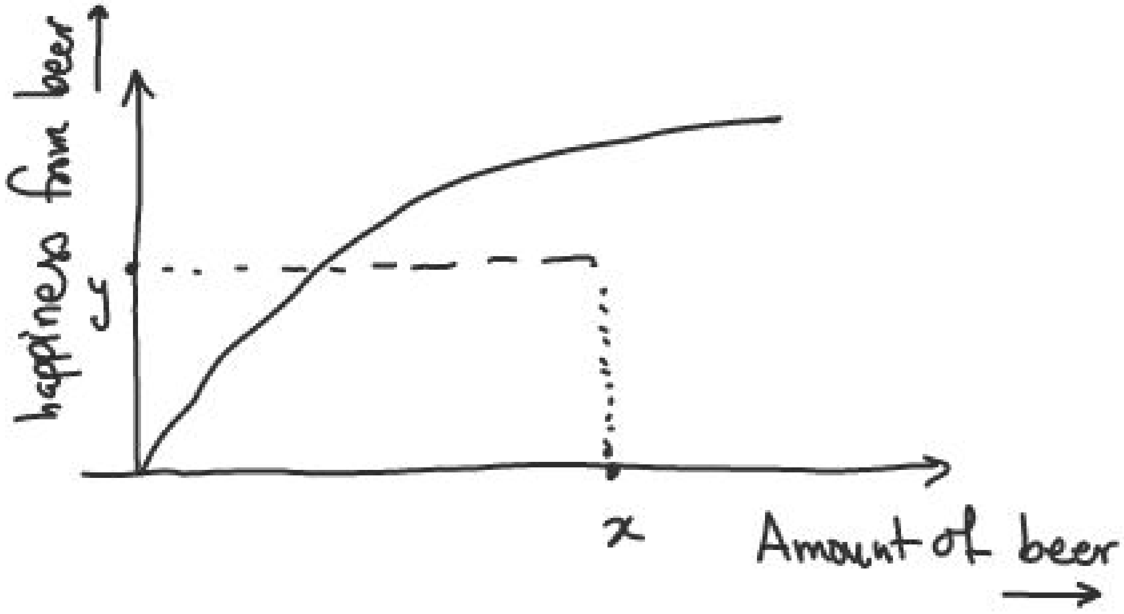


Normals

* **Note that:** If $x \in C$ interior, then $N(x) = \bar{0}$. Thus if x minimizes c then $\nabla c(x) = \bar{0}$, $\frac{\partial c}{\partial x_i} = 0 \ \forall i$

Example: $D = \{(x, y) \mid x \geq 0,\ y \geq 0,\ y \leq u(x)\}$

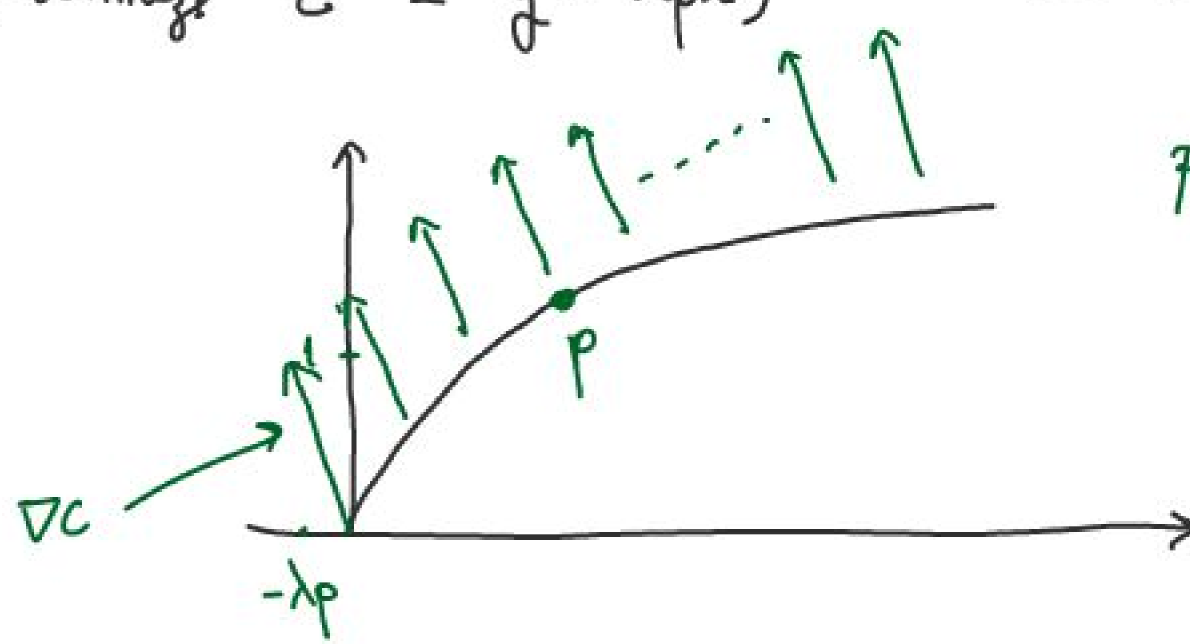
$y \leq \sqrt{x}$



price of beer is p

value of a rupee is λ units

(Maximize $c = y - \lambda p x$) $\quad \nabla c = [-\lambda p, 1]$



p: is a point where ∇c is normal to $y \leq \sqrt{x}$ fn.

$$y - u(x) \leq 0 \qquad \nabla(y - u(x)) = \left[\frac{\partial u}{\partial x}, 1\right]$$

$$\text{cone}(p) = \lambda \left[\frac{\partial u}{\partial x}, 1\right]$$

Optimality condit'n : $\lambda = \dfrac{\partial u / \partial x}{p}$

- if λ is small (do not value money) then point p will occur much later on the curve.
- On the other hand if λ is large (ie. value of money is more) then point p will occur early on the curve.