Tutorial 1

1. Let $X_1, X_2, \ldots, X_m$ and $Y_1, \ldots, Y_m$ be two finite collection of sets so that $X_i \cap Y_i = \phi$ for all i. Show that $\cup_i X_i$ and $\cap_i Y_i$ are disjoint.

2. Let C_α for $\alpha \in I$ be an arbitrary collection of closed sets. Show that $\cap_\alpha C_\alpha$ is also closed. Show by an example, that it may also be empty.

3. We say that a downward or upward ideal I is *principal* if it is generated by a single element, i.e, $I = (x \downarrow) = \{y | y \leq x\}$, when I is downward closed, and $I = (x \uparrow) = \{y | y \geq x\}$ if I is upward closed. Let $s \subset \mathbb{R}$ be an upward ideal, i.e., $x > y$ and $y \in s$ implies that $x \in s$. Let r be its complement in $\mathbb{R}$. When can r and s be principal/not principal?

4. Let S_1 and S_2 be metric spaces. Can $S = S_1 \times S_2$ be made into a metric space? What would be the open/closed sets in S?

5. Consider the various metrics on $\mathbb{R}^2$ and show that the topologies generated by them are identical, i.e., the collection of open sets are the same.

6. Let $p(x)$ be a polynomial with roots $\alpha_1, \ldots, \alpha_k$. Let $E(p) = \{x \in \mathbb{R} | p(x) < 0\}$. Is E open? What are the limit points of E? What is the infimum and the supremum?

7. Let E be any set in a metric space S. For an $\epsilon > 0$, define $C_\epsilon(E) = \{y | d(x, y) \leq \epsilon$ for some $x \in E\}$. If E is a closed set, then is $C_\epsilon(E)$ closed? What if E is open?

   Let $E \subset \mathbb{R}$. For an open finite set of open intervals $I = \{(a_i, b_i)\}_i$ which covers E, define the I-length of E, $\ell_I(E) = \sum_i (b_i - a_i)$. In other words, it is an estimate of the length of E by using an open cover. Now, define $\ell(E) = \inf\{\ell_I(E)\}I$, i.e., an infimum of the lengths of all possible open covers of E. Show that (i) $\ell([0, 1]) = \ell((0, 1)) = 1$. Let E be the collection of all rationals in $[0, 1]$. What is $\ell(E)$?