Quantum deformations of the restriction of some $GL_{mn}(\mathbb{C})$-modules to $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$

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Talk Outline

- The GCT perspective and the $G \longrightarrow G \times G$ case.
- The $U_q(gl_m) \otimes U_q(gl_n)$ structure of $V_\lambda(\mathbb{C}^{mn})$ for some λ.
  - The structure on $\wedge^k(\mathbb{C}^{mn})$.
  - The bi-crystal structure on $\wedge^k(\mathbb{C}^{mn})$.
  - The straightening laws and the general case.
- An m-crystal structure for $V_\lambda(\mathbb{C}^{m \cdot 2})$.
- Conclusion.

The Perspective

- The Key: The determination of Peter-Weyl modules for the pair (H, G) with $H \subseteq G$.
  - When does $V_\lambda(G)$ have an H-fixed vector.
  - A conceptual and effective answer.
- H is typically a reductive group, a stabilizer of a stable form.
- The special case being the $det(X)$ where $GL_m \times GL_m \rightarrow GL_{m^2}$ given by:

$$(A, B)(X) \rightarrow AXB^{-1}$$

- For this talk, the more general $GL_m \times GL_n \rightarrow GL_{mn}$.

The $G \to G \times G$ case: Combinatorics

Largely, the GL_m-case.

- $SS(\lambda, m)$, column-strict semi-standard tableau of shape λ with entries in $[m]$.
- The monoid $M(m)$ of words on $[m]$ and the Plactic Monoid $PM(m)$.
- The row-bump operation and the map $M(m) \to PM(m)$.

$$\begin{array}{|c|c|c|c|c|}\hline 3 & 2 & 4 & 2 & 4 \\ \hline \end{array} \xrightarrow{RSK} \begin{array}{|c|c|c|}\hline 2 & 2 & 4 \\ \hline 3 & 4 & \\ \cline{1-2} \end{array} \in SS([3,2],4)$$

- *jeu-de-taquin* for multiplying two tableau:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & & \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 4 & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 3 & 3 & 4 & \\ \cline{1-3} \end{array}$$

The Bridge

- The connection between $V_\lambda(GL_m)$ and $SS(\lambda, m)$.
  - At the weight-space level

$$dim(V(\lambda)[\mu]) = |SS(\lambda, m)[\mu]|$$

- Moreover, at the tensor-product level

$$SS(\lambda, m) \times SS(\square, m) \equiv V_\lambda(GL_m) \otimes V_\square(GL_m)$$

- More generally,

$$SS(\lambda, m) \times SS(\mu, m) \equiv V_\lambda(GL_m) \otimes V_\mu(GL_m)$$

The Algebra

- The Drinfeld-Jimbo algebra $U_q(gl_m)$ and the Hopf:

$$\Delta : U_q(gl_m) \to U_q(gl_m) \otimes U_q(gl_m)$$

- Date-Jimbo-Miwa explanation of the row-bump and RSK.
- The Kashiwara-Lusztig crystal base and various models.
  - identification of $SS(\lambda, m)$ with specific basis elements in $V_\lambda(\mathbb{C}^m)$.
- The Kashiwara tensor product rule.

Moreover, much of the theory worked beyond GL_m.

Richer combinatorics

- Crystal Operators $\mathcal{E}_i, \mathcal{F}_i$ on $M(m), PM(m)$, i.e., on words and tableaus $SS(\lambda, m)$, e.g., E_3:

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&4\\\hline 2&3&4\\\cline{1-3} 4\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1&2&2&4\\\hline 2&3&*\\\cline{1-3} 4\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1&2&2&4\\\hline 2&3&3\\\cline{1-3} 4\\\cline{1-1}\end{array}$$

- Our interest: Littlewood-Richardson coefficients:

$$V_\lambda \otimes V_\mu = \oplus_\beta c^\beta_{\lambda,\mu} V_\beta$$

- Proofs of the PRV and LR rule.
- The Berenstein-Zelevinsky polytope model: $c^\beta_{\lambda,\mu}$ as integer points in a suitable polytope.

Finally..

- The Knutson-Tao Hive model.
- The saturation conjecture proved:

$$c^{n\beta}_{n\lambda,n\mu} > 0 \Rightarrow c^{\beta}_{\lambda,\mu} > 0$$

- Abstract polynomial time algorithm to detect if $c^{\beta}_{\lambda,\mu} > 0$.
- Burgisser: A simple algorithm.

Finally..

- The Knutson-Tao Hive model.
- The saturation conjecture proved:

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- Abstract polynomial time algorithm to detect if $c_{\lambda,\mu}^{\beta} > 0$.
- Burgisser: A simple algorithm.

Conclusion: conceptual and effective

Th quantum algebra route has settled the Peter-Weyl problem for

$$GL_m \rightarrow GL_m \times GL_m$$

i.e., a simple algorithm to detect if $c_{\lambda,\mu}^{\beta} > 0$.

The $GL_m \times GL_n \to GL_{mn}$-case: mainly RSK

.

- $Sym(m,n)$: collection of $m \times n$ matrices with $\mathbb{Z}^+$ entries.
$$Sym(m,n) \to \cup_\lambda SS(\lambda,[m]) \times SS(\lambda,[n])$$
- $Wedge(m,n)$: collection of $m \times n$ matrices with 0-1 entries.
$$Wedge(m,n) \to \cup_\lambda SS(\lambda,[m]) \times SS(\lambda^T,[n])$$
- Both these match the module and weight-space decompositions for $Sym^k(\mathbb{C}^{mn})$ and $\wedge^k(\mathbb{C}^{mn})$.

1	0	0	1
0	1	0	1
1	1	1	0

$LW(b) = 3132321$ $\quad RW(b) = 3214241$

$$LT(b) = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 3 & & \\ \hline \end{array} \qquad RT(b) = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 2 & \\ \hline 3 & 4 & \\ \hline \end{array}$$

Recently..

Danilov and Koshevoi, van Leeuwen:

- Constructed a *combinatorial* bi-crystal-graph structure on $Sym(m,n)$ and $Wedge(m,n)$.
- $\mathcal{E}_i^L, \mathcal{F}_i^L$ for $i = 1, \ldots, m-1$ and $\mathcal{E}_j^R, \mathcal{F}_j^R$ for $j = 1, \ldots, n-1$.

No other general case is known. Also not known:

- algebraic basis for Danilov's operators.
- A quantization

$$U_q(gl_m) \otimes U_q(gl_n) \to U_q(gl_{mn})$$

This may not even exist.., see Hayashi. The injection $U_1(gl_m) \otimes U_1(gl_n) \to U_1(gl_{mn})$ is straight-forward.

We construct ...

- an embedding $U_q(gl_m) \otimes U_q(gl_n) \to U_q(gl_{mn})$ on the module $\wedge^k(\mathbb{C}^{mn})$, i.e.,

$$U_q(gl_{mn}) \longrightarrow End_{\mathbb{C}[q,q^{-1}]}(\wedge^k(\mathbb{C}^{mn})) \longleftarrow U_q(gl_m) \otimes U_q(gl_n)$$

- A bi-crystal basis for $\wedge^k(\mathbb{C}^{mn})$.
- First, for 2-column λ, a $U_q(gl_m) \otimes U_q(gl_n)$-module W_λ such that at $q = 1$, the module is isomorphic to $V_\lambda(\mathbb{C}^{mn})$ restricted to $U_1(gl_m) \otimes U_1(gl_n) \subseteq U_1(gl_{mn})$.
- Possible straightening laws.

Notation

We have $N = mn$ and the symbols e_i, f_i for $i = 1, \ldots, N-1$ and $q^{\epsilon_i}, q^{-\epsilon_i}$, for $i = 1, \ldots, N$ so that:

$$q^{\epsilon_i} q^{-\epsilon_i} = q^{-\epsilon_i} q^{\epsilon_i} = 1, \quad [q^{\epsilon_i}, q^{\epsilon_j}] = 0$$

$$q^{\epsilon_i} e_j q^{-\epsilon_i} = \begin{cases} q e_j & \text{for } i = j \\ q^{-1} e_j & \text{for } i = j+1 \\ e_j & \text{otherwise} \end{cases}$$

$$q^{\epsilon_i} f_j q^{-\epsilon_i} = \begin{cases} q^{-1} f_j & \text{for } i = j \\ q f_j & \text{for } i = j+1 \\ f_j & \text{otherwise} \end{cases}$$

We also use $q^{h_i} = q^{\epsilon_i} q^{-\epsilon_{i+1}}$ and $q^{-h_i} = q^{-\epsilon_i} q^{\epsilon_{i+1}}$.

More ...

- The brackets

$$[e_i, f_j] = \delta_{ij} \frac{q^{\epsilon_i} q^{-\epsilon_{i+1}} - q^{-\epsilon_i} q^{\epsilon_{i+1}}}{q - q^{-1}}$$

$$[e_i, e_j] = [f_i, f_j] = 0 \text{ for } |i - j| > 1$$

- The braids

$$e_j e_i^2 - (q + q^{-1}) e_i e_j e_i + e_i^2 e_j = f_j f_i^2 - (q + q^{-1}) f_i f_j f_i + f_i^2 f_j = 0$$

when $|i - j| = 1$.

- The Hopf

$$\Delta q^{\epsilon_i} = q^{\epsilon_i} \otimes q^{\epsilon_i}$$
$$\Delta e_i = e_i \otimes 1 + q^{-h_i} \otimes e_i, \Delta f_i = f_i \otimes q^{h_i} + 1 \otimes f_i$$

Our model

We model $\wedge^k(\mathbb{C}^{mn})$ as the vector space with basis as the k-subsets $c \subseteq [mn]$. For a set c, we denote v_c as the basis element.

$$q^{\epsilon_i} v_c = \begin{cases} v_c & \text{if } i \notin c \\ q v_c & \text{otherwise} \end{cases}$$

$$e_i v_c = \begin{cases} 0 & \text{if } i+1 \notin c \text{ or } i \in c \\ v_d & \text{otherwise, where } d = c - \{i+1\} + \{i\} \end{cases}$$

$$f_i v_c = \begin{cases} 0 & \text{if } i+1 \in c \text{ or } i \notin c \\ v_d & \text{otherwise, where } d = c - \{i\} + \{i+1\} \end{cases}$$

Thus e_i drops $i+1$ from c and introduces an i, whenever it can be done. Similarly f_i.

On the wedges...

- $e_i^2 = 0$, $e_i f_{i+1} = e_{i+1} f_i = 0$ for all i.
- $e_i e_j e_i = 0$ whenever $|j - i| = 1$.

For $i < j$, let $E_{i,j}$ denote the term $[e_i, [e_{i+1}, [\ldots [e_{j-1}, e_j]]]]$ and $F_{i,j}$ denote $[[[f_j, f_{j-1}], \ldots, f_i]]$. *Note the ordinary bracket.*

$$E_{i,j}(v_c) = \begin{cases} (-1)^{|c \cap [i+1,j]|} v_d & \text{if } j+1 \in c \text{ and } i \notin c \\ & \text{where } d = c - \{j+1\} + \{i\} \\ 0 & \text{otherwise} \end{cases}$$

Note the jumping count and the *sign*. Thus:

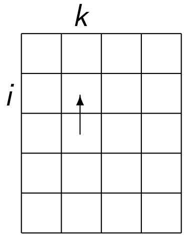
$$E_{1,3}\left(\begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} \right) = - \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$$

The embedding

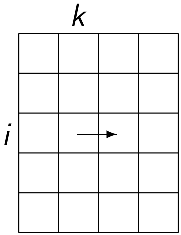
We identify $\mathbb{C}^{m \cdot n}$ with $\mathbb{C}^{m \otimes n}$:

1	4	7	10
2	5	8	11
3	6	9	12

We will mainly use the following as basic operators:



$e_{(k-1)m+i}$



$F_{(k-1)m+i,km+i-1}$

$U_q^L(gl_m)$ and $U_q^R(gl_n)$

We will now define the left operators E_i^L, F_i^L and $q^{\epsilon_i^L}$ and the right operators E_j^R, F_j^R and $q^{\epsilon_j^R}$. It is clear that:

$$q^{\epsilon_i^L} = \prod_{j=0}^{n-1} q^{\epsilon(mj+i)} \qquad q^{\epsilon_j^R} = \prod_{i=1}^{m} q^{\epsilon(m(j-1)+i)}$$

Pictorially:

$q^{\epsilon_2^L} =$

0	0	0	0
1	1	1	1
0	0	0	0

$q^{\epsilon_3^R} =$

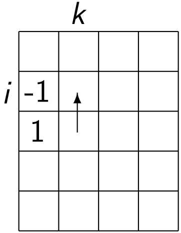
0	0	1	0
0	0	1	0
0	0	1	0

The left operators

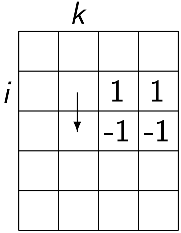
Next, we define the **left operators** using:

$$
\begin{aligned}
B_i^k &= \textstyle\sum_{j=0}^{k-2} -h_{jm+i} \\
A_i^k &= \textstyle\sum_{j=k}^{n-1} h_{jm+i}
\end{aligned}
$$

$$
\begin{aligned}
E_i^L &= q^{B_i^1} e_i + q^{B_i^2} e_{m+i} + \ldots q^{B_i^n} e_{(n-1)m+i} \\
F_i^L &= q^{A_i^1} f_i + \ldots + q^{A_i^{n-1}} f_{(n-2)m+i} + q^{A_i^n} f_{(n-1)m+i}
\end{aligned}
$$



$q^{B_i^k} e_{(k-1)m+i}$



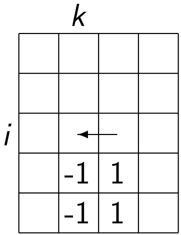
$q^{A_i^k} f_{(k-1)m+i}$

The right operators

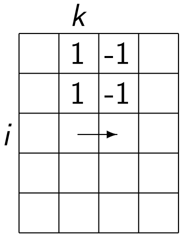
We define the **right operators** using:

$$\beta_i^k = \sum_{j=i+1}^{m} \epsilon_{km+j} - \sum_{j=i+1}^{m} \epsilon_{(k-1)m+j}$$
$$\alpha_i^k = \sum_{j=1}^{i-1} \epsilon_{i(k-1)m+j} - \sum_{j=1}^{i-1} \epsilon_{km+j}$$

$$E_k^R = \sum_{i=1}^{m} q^{\beta_i^k} E_{(k-1)m+i,km+i-1}$$
$$F_k^R = \sum_{i=1}^{m} q^{\alpha_i^k} F_{(k-1)m+i,km+i-1}$$

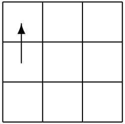


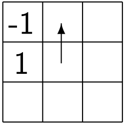
$q^{\beta_i^k} E_{(k-1)m+i,km+i-1}$

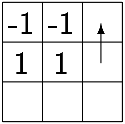


$q^{\alpha_i^k} F_{(k-1)m+i,km+i-1}$

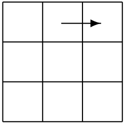
A small example: Let $m = n = 3$

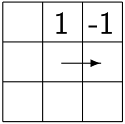


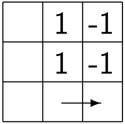




$$e_1 \quad + \quad q^{\epsilon_2-\epsilon_1} e_4 \quad + \quad q^{\epsilon_2+\epsilon_5-\epsilon_1-\epsilon_4} e_7 \quad = E_1^L$$







$$F_{4,6} \quad + \quad q^{\epsilon_4-\epsilon_7} F_{5,7} \quad + \quad q^{\epsilon_4+\epsilon_5-\epsilon_7-\epsilon_8} F_{6,8} \quad = F_2^R$$

So then...

- The left operators do treat the matrix as a tensor of columns, left to right.
- The right operators treat the matrix as a tensor of row, bottom to top and with a sign.

Check

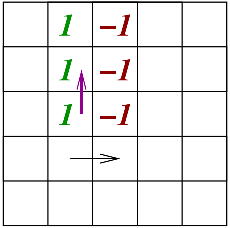
- Check that $\{E_i^L, F_i^L, q^{\epsilon_i^L}\}$ together satisfy the properties for $U_q(gl_m)$.
- Same for $\{E_k^R, F_k^R, q^{\epsilon_k^R}\}$ and $U_q(gl_n)$.
- That these two actions commute on $\wedge^k(\mathbb{C}^{mn})$.

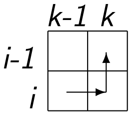
Remarks

- Actually, the left $U_q(gl_m)$ comes from:

$$U_q(gl_m) \xrightarrow{\Delta} U_q(gl_m) \otimes \ldots \otimes U_q(gl_m) \to U_q(gl_{mn})$$

  Thus, it is actually sitting inside $U_q(gl_{mn})$.
- The right copy has no analogue in $U_q(gl_{mn})$ and is synthetic. But for the sign, the action is similar.
- The commutation reduces to sl_2-sl_2 case, is a calculation.





Moreover

- We may check that at $q = 1$ the action matches the injection $U_1(gl_m) \otimes U_1(gl_n) \to U_1(gl_{mn})$.
- This implies that $\wedge^k(\mathbb{C}^{mn})$ is isomorphic to $\oplus_\lambda V_\lambda(\mathbb{C}^m) \otimes V_{\lambda^T}(\mathbb{C}^n)$ as $U_q(gl_m) \otimes U_q(gl_n)$-modules.
- In fact, the highest weight vectors v_λ are those from subsets c_λ in the upper left corner of the shape λ.

$$c_{(3,1)} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad v_{(3,1)} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 7 \\ \hline \end{array}$$

- Thus $\wedge^k(\mathbb{C}^{mn})$ as a $U_q(gl_m) \otimes U_q(gl_n)$-module has been constructed.

Next, the crystal base for $\wedge^k(\mathbb{C}^{mn})$

For a subset $c \subseteq [mn]$, let v_c denote the *pure* element in $\wedge^k$. Then, there is a $sign(c)$ such that the set

$$\{sign(c) \cdot v_c | c \subseteq [mn], |c| = k\}$$

is the crystal base for $\wedge^k(\mathbb{C}^{mn})$.

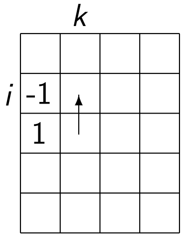
- Let $U_i^L \equiv U_q(sl_2)$ be the algebra generated by $E_i^L, F_i^L, q^{h_i^L}$.
- For a subset $c \subseteq [mn]$, let $V_L(c)$ be the vector space generated by all subsets c' which match c in the column-sums for the rows $i, i+1$ and c matches c' everywhere else.

i	1	1	1	1		1	1
i+1		1		1			1

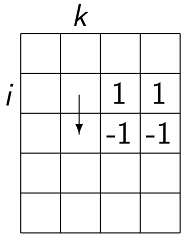
i		1	1	1			1
i+1	1	1		1		1	1

At once!

$V_L(c)$ is U_i^L equivariant, and is of dimension 2^k for some k. In fact, $V_L(c)$ is isomorphic to $\otimes^k V_{(1)}$.



$q^{B_i^k} e_{(k-1)m+i}$



$q^{A_i^k} f_{(k-1)m+i}$

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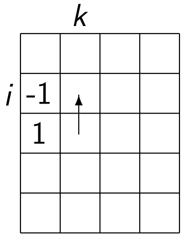

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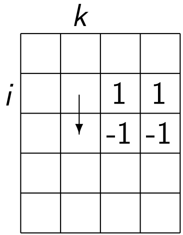


$q^{A_i^k} f_{(k-1)m+i}$

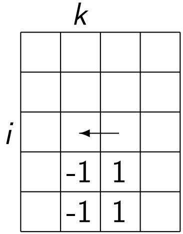
Note that inactive columns dont add a q-factor!

- It follows that the pure elements constitute a crystal basis for the left action.
- The crystal operator $\mathcal{E}_i^L$ is also clear!

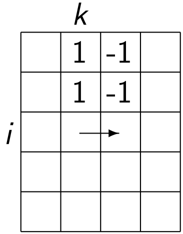
Now the right

Two complications

- The Hopf works the other way
- There are signs!



$q^{\beta_i^k} E_{(k-1)m+i,km+i-1}$



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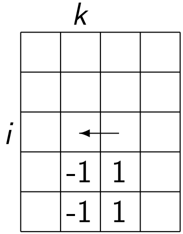

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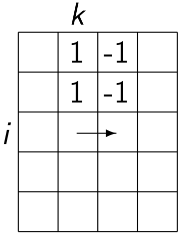


$q^{\alpha_i^k} F_{(k-1)m+i,km+i-1}$

- Tricky. Define a local sign for each U_i^R so that:

$$E_{(k-1)m+i,km+i-1} v_c = sign(d)/sign(c) v_d$$

- Define a global sign which is consistent only on the crystal operators. For $m = n = 2$ $sign(\{2,3\}) = -1$, all others $+1$

The combinatorics

How do we implement $\wedge^*(\mathbb{C}^{mn}) \leftrightarrow \cup_\lambda SS(\lambda, m) \times SS(\lambda', n)$?

The two Hopfs give us the reading order:

- left: read columns bottom to top, left to right.
- right: read row back to front, bottom to top.

Let $m = 3$ and $n = 4$ and let $b = \{1, 3, 5, 6, 9, 10, 11\}$.

1	0	0	1
0	1	0	1
1	1	1	0

$LW(b) = 3132321$ $\qquad RW(b) = 3214241$

$LT(b) =$

1	1	2
2	3	3
3		

$RT(b) =$

1	1	4
2	2	
3	4	

Question: How do I compute b from $LT(b), RT(b)$?

Towards the general module

- The algebra $U_q(gl_m) \otimes U_q(gl_n)$ comes with a Hopf, whence $\wedge^a(\mathbb{C}^{mn}) \otimes \ldots \otimes \wedge^z(\mathbb{C}^{mn})$ are all available.
- seems difficult to identify $V_\lambda(\mathbb{C}^{mn})$ as a submodule.
- We construct equivariant injections and the 2-column modules

$$\psi_{a,b} : \wedge^{a+1} \otimes \wedge^{b-1} \longrightarrow \wedge^a \otimes \wedge^b$$

- The images of $R_{a,b}$ are the straightening laws .

$$\mathcal{S} = \{R_{a,b} | 1 \leq b \leq a \leq mn\}$$

- We **faintly hope** that, if $\lambda^T = [a_1, \ldots, a_k]$ then

$$V_\lambda(\mathbb{C}^{mn}) = \wedge^{a_1} \otimes \ldots \otimes \wedge^{a_k} / \mathcal{S}$$

The ψ's

- Let $\mu : \wedge^a \otimes \wedge^b \longrightarrow \wedge^c$, equivariant.
- Let $[\mu]$ be the matrix of the map in the *standard basis* of sets.
- Then $[\mu]^T : \wedge^c \longrightarrow \wedge^a \otimes \wedge^b$ is also equivariant and good.

We use this to construct merely:

$$L_a : \wedge^a \longrightarrow \wedge^1 \otimes \wedge^{a-1}$$
$$R_a : \wedge^a \longrightarrow \wedge^{a-1} \otimes \wedge^1$$

We obtain $\psi_{a,b}$ as the composition:

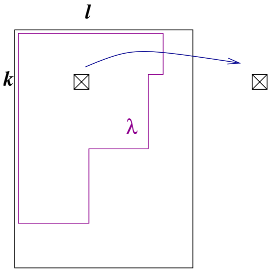
$$\wedge^{a+1} \otimes \wedge^{b-1} \xrightarrow{R_{a+1} \otimes id} \wedge^a \otimes \wedge^1 \otimes \wedge^{b-1} \xrightarrow{id \otimes [L_b]^T} \wedge^a \otimes \wedge^b$$

The L_a and R_a

- $\wedge^a(\mathbb{C}^{mn})$ is multiplicity-free and we have the highest weight subset c_λ and v_λ.
- $\wedge^{a-1} \otimes \wedge^1$ is not multiplicity-free!
- We will define R_a and L_a only for these v_λ and extend it.
- Furthermore, at $q = 1$, the map $L_a(v_\lambda)$ and $R_a(v_\lambda)$ will match the classical $U_1(gl_{mn})$-expressions.

For a shape λ sitting inside $m \times n$,

- let $c_{kl} = c_\lambda - (k, l)$ and $t_{kl} = v_{c_{kl}} \in \wedge^{a-1}(\mathbb{C}^{mn})$.
- χ_{kl} be the vector $v_{(k,l)} \in \wedge^1(\mathbb{C}^{mn})$



The vectors

Here is R_a:

$$R_a(v_\lambda) = \sum_{(k,l)\in\lambda} \alpha_{kl} t_{kl} \otimes \chi_{kl}$$

$$\in \wedge^{a-1} \otimes \wedge^1$$

$$\alpha_{kl} = (-1)^{\lambda'_1+\ldots+\lambda'_{l-1}+k} q^{k+l-\lambda_k}$$

And here is L_a:

$$L_a(v_\lambda) = \sum_{(k,l)\in\lambda} \beta_{kl} \chi_{kl} \otimes t_{kl}$$

$$\in \wedge^1 \otimes \wedge^{a-1}$$

$$\beta_{kl} = (-1)^{\lambda'_1+\ldots+\lambda'_{l-1}+k} q^{\lambda'_l-k-l}$$

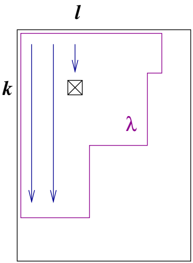
The vectors

Here is R_a:

$$R_a(v_\lambda) = \sum_{(k,l)\in\lambda} \alpha_{kl} t_{kl} \otimes \chi_{kl} \in \wedge^{a-1} \otimes \wedge^1$$

$$\alpha_{kl} = (-1)^{\lambda'_1+\ldots+\lambda'_{l-1}+k} q^{k+l-\lambda_k}$$

What happens at $q = 1$?



And here is L_a:

$$L_a(v_\lambda) = \sum_{(k,l)\in\lambda} \beta_{kl} \chi_{kl} \otimes t_{kl} \in \wedge^1 \otimes \wedge^{a-1}$$

$$\beta_{kl} = (-1)^{\lambda'_1+\ldots+\lambda'_{l-1}+k} q^{\lambda'_l-k-l}$$

This proves the *mn*-equivariance at $q = 1$, and thus the construction of $V_\lambda(\mathbb{C}^{mn})$ for 2-columns.

Straighten too much?

- Recall

$$\psi_{a,b} : \wedge^{a+1} \otimes \wedge^{b-1} \longrightarrow \wedge^{a} \otimes \wedge^{b}$$

- That $\psi_{a,b}$ is an injection implies that $\mathcal{S}$ cannot straighten too little.
- So the only issue with

$$V_\lambda(\mathbb{C}^{mn}) = \wedge^{a_1} \otimes \ldots \otimes \wedge^{a_k} / \mathcal{S}$$

is that it may straighten too much.

- Our $\psi_{a,b}$ at $q = 1$ is $U_1(gl_{mn})$-equivariant and matches the standard straightening laws.
- Does this proves the construction? NOT YET
- True for *Sym*!

Remarks

- Only at v_λ is the $[mn]$-weight preserved. For $m = n = 2$, $(q^2+1)\cdot R_2(\{2,3\})$ is:

  $$(q^3-1)/q\cdot 1\otimes 4-(q+1)\cdot 2\otimes 3+(q+1)\cdot 3\otimes 2+(q-1)\cdot 4\otimes 1$$

- We have achieved:

  $$\wedge^a \overset{\longleftarrow}{\longrightarrow} \wedge^{a-1}\otimes\wedge^1 \overset{\longleftarrow}{\longrightarrow} \dots \wedge^r\otimes\wedge^s \overset{\longleftarrow}{\longrightarrow} \dots \wedge^1\otimes\wedge^{a-1} \overset{\longleftarrow}{\longrightarrow} \wedge^a$$

Perhaps, R_a, L_b can be so chosen so that an additional $U_q(gl_m)\otimes U_q(gl_n)\otimes U_q(gl_2)$ structure on $\wedge^a(\mathbb{C}^{2mn})$ is established!

Moreover...

This reduces to finding, say $\{R_a\}_a$ such that:

$$EF_a : \wedge^{a-1} \otimes \wedge^1 \overset{\longleftarrow}{\longrightarrow} \wedge^{a-2} \otimes \wedge^1 \otimes \wedge^1 \overset{\longleftarrow}{\longrightarrow} \wedge^{a-2} \otimes \wedge^2$$

has TWO eigenvalues.

If such local maps are found then we have obtained a $U_q(gl_m) \otimes U_q(gl_n) \otimes U_q(gl_2)$ structure on $\wedge^*(\mathbb{C}^{mn2})$.

The general $U_q(gl_m) \otimes U_q(gl_n) \otimes U_q(gl_p)$ on $\wedge^k(\mathbb{C}^{mnp})$ will side-step the straightening laws.

Indeed..

On hind-sight $U_q(gl_m) \otimes U_q(gl_2)$ is obvious!

$$EF_a : \wedge^{a-1} \otimes \wedge^1 \overset{\longleftarrow}{\longrightarrow} \wedge^{a-2} \otimes \wedge^1 \otimes \wedge^1 \overset{\longleftarrow}{\longrightarrow} \wedge^{a-2} \otimes \wedge^2$$

- $n = 1$ implies $\wedge^a \otimes \wedge^1$ is multiplicity free with two irreducibles.
- In fact, our right operators are in this "factored" format.

These right-operators are essentially raising and lowering operators:

$$\wedge^{[a_1,\dots,a_i,a_{i+1},\dots,a_n]}(\mathbb{C}^m) \xrightarrow{E_i^R} \wedge^{[a_1,\dots,a_i-1,a_{i+1}+1,\dots,a_n]}(\mathbb{C}^m)$$

which are factored and local and satisfy the Serre relations.

The Big Picture

Obviously, get the left and right operators on $V_\lambda(\mathbb{C}^{mn})$.

But there are many paths to it:

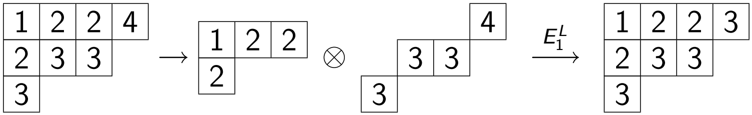
- Unwind the straightening laws.
- Get $U_q(gl_m) \otimes U_q(gl_n) \otimes U_q(gl_r)$ structure on $\wedge^k(\mathbb{C}^{mnr})$ and get its crystal base.
  - The $2mn$ case is already novel: Young poset
- A Hecke-type operator on $\wedge^1(\mathbb{C}^{mn}) \otimes \wedge^1(\mathbb{C}^{mn})$ commuting with the action of $U_q(gl_m) \otimes U_q(gl_n)$? GCT4 with Ketan.
- some other way?

A Puzzle

Lets consider the case of $GL_m \to GL_{2m}$-the block diagonal embedding. We also have:

$$U_q(gl_m) \xrightarrow{\Delta} U_q(gl_m)[1, m-1] \otimes U_q(gl_m)[m+1, 2m-1] \to U_q(gl_{2m})$$

This gives us a $U_q(gl_m)$-structure on $V_\lambda(\mathbb{C}^{2m})$.

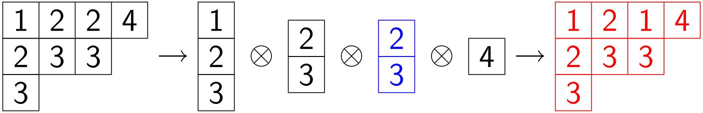


In other words

$$E_1^L = e_1 \otimes e_3$$

Tempting to seek E_1^R as a tensor of some existing $2m$-operators, maybe after some Weyl group action. That fails.

But even for the left operator ...



Thus, the column-wise tensor does not hold!

But even for the left operator ...

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&4\\\hline 2&3&3\\\cline{1-3} 3\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 4\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1&2&1&4\\\hline 2&3&3\\\cline{1-3} 3\\\cline{1-1}\end{array}$$

Thus, the column-wise tensor does not hold! Here is the **magic massage**:

$$\begin{array}{|c|c|c|c|}\hline 1&2&1&4\\\hline 2&3&3\\\cline{1-3} 3\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1&1&1&4\\\hline 2&3&4\\\cline{1-3} 3\\\cline{1-1}\end{array}$$

One may indeed define an m-crystal structure on $SS(\lambda, 2m)$ which

- works column-wise and acts at the right place.
- massages in a structured way, only the columns on the left.

Is this a fragment of the crystallization?

The $m = n = 2$ case, any shape.

The left operators go left-to-right. The right operators go down.

$$
\begin{array}{ccccc}
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 4 \\ \hline\end{array} & \longrightarrow & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline\end{array} & \longrightarrow & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & 4 \\ \hline\end{array} \\
\downarrow & & \downarrow & & \downarrow \\
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 4 \\ \hline\end{array} & \longrightarrow & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array} & \longrightarrow & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 4 \\ \hline\end{array} \\
\downarrow & & \downarrow & & \downarrow \\
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline\end{array} & \longrightarrow & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array} & \longrightarrow & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}
\end{array}
$$

Does this picture have a quantum explanation :

Thank you.