Exchange Markets: Strategy meets Supply-Awareness

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Abstract. Market equilibrium theory assumes that agents are truthful, and are generally unaware of the total supply of goods in the market. In this paper, we study *linear* exchange markets with each of these assumptions dropped separately, and show a surprising connection between their solutions.

We define the exchange market game as where agents strategize on their utility functions, and we derive a complete characterization of its symmetric Nash equilibria (SNE). Using this characterization we show that the payoffs at SNE are Pareto-optimal, the SNE set is connected, and we also obtain necessary and sufficient conditions for its uniqueness.

Next we consider markets with supply-aware agents, and show that the set of competitive equilibria (CE) of such a market is equivalent to the set of SNE of the corresponding exchange market game. Through this equivalence, we obtain both the welfare theorems, and connectedness and uniqueness conditions of CE for the supply-aware markets.

Finally, we extend the connection between CE and SNE to exchange markets with arbitrary concave utility functions, by restricting strategies of the agents to linear functions in the game, and as a consequence obtain both the welfare theorems.

1 Introduction

General equilibrium theory has been studied extensively for more than a century due to its immense practical relevance [15,29]. The exchange market model is a classical market model proposed by Leon Walras in 1874 [34]. In this model, each agent has a fixed initial endowment of goods, which she can sell and buy a preferred bundle of goods from her earned money. Her utility for a bundle of goods is determined by a non-decreasing, concave function. Given prices of goods, each agent demands a utility maximizing (optimal) bundle, that is affordable by her earned money. A setting of prices is referred to as competitive equilibrium (CE) if, after each agent is given an optimal bundle, there is neither deficiency nor surplus of any good, *i.e.,* the market clears. It was only in 1954, that Arrow and Debreu showed the existence of a competitive equilibrium [3][3] under mild

[3] They consider a more general model including production firms.

conditions. Since then, there has been a large body of work to understand the properties, structure and consequences of competitive equilibrium [22,31].

In this market model it is implicitly assumed that agents behave truthfully, and are unaware of the total supply of goods available in the market. Each of these assumptions may not necessarily hold as observed and analyzed for different market settings [1,5,7,8]. In this paper we study exchange markets, with each of these assumptions dropped separately, and establish a surprising connection between their solutions which we think should be of economic interest.

The strategic behavior of agents is well known; many different types of market games have been formulated and analyzed for its Nash equilibria by economists [2,13,28] (see Section 1.2 for details). More recently, [1] defined the Fisher market[4] game for linear utility functions where agents strategize on their utility functions, and they derived various properties of its Nash equilibria. Further, [7] showed that no agent can gain more than twice by strategizing in Fisher markets with linear utility functions; a similar result is obtained for Fisher markets with Leontief utility functions in [8]. To the best of our knowledge no such results are known for the exchange markets.

Generalizing the Fisher market game of [1], we define the *exchange market game*, as where agents are players and strategies are utility functions that they may pose, for the case when utility functions are linear. We derive a complete characterization of the symmetric Nash equilibria (SNE) this game.

In strategic analysis of markets, a crucial question is whether competitive equilibrium allocations, which are always efficient, can be achieved at Nash equilibrium [28,2,13], and even better if no Nash equilibrium is sub-optimal. Using the characterization of SNE we obtain a number of such important properties: (*i*) the payoffs at SNE are always Pareto-optimal, and (*ii*) every CE allocation can be achieved at a SNE. Apart from these, we obtain structural properties for the SNE set, like (*iii*) connectedness, and (*iv*) necessary and sufficient conditions for uniqueness. These structural properties are quite sought after in equilibrium theory, both competitive and Nash, and a lot of work has been done to characterize such instances [26,25,21,16].

For the case of arbitrary concave utilities, we derive sufficiency conditions for a strategy to be a symmetric Nash equilibrium, and obtain the first two properties of Pareto-optimality, and achieving CE allocations at SNEs. We note that the analysis in exchange market game is relatively more involved than in Fisher, as expected.

The other assumption that agents are unaware of the total supply of goods in the market, may not hold in many rural and informal markets where supplies are visible. Given that agents know the supply of all the goods, it is rational for them to take the supplies in to consider while calculating their demand bundles. Therefore, the dynamics of demands will change which in turn will change the set of competitive equilibria. Such a setting has been analyzed for auction markets, where agents are assumed to be *supply-aware* in finding the equilibrium [5,6].

[4] Fisher market is a special case of exchange market model [33].

However, no such work exploring equilibria of exchange markets with supply-aware agents is known.

We make significant progress towards understanding the effect of supply-aware agents in exchange markets. We obtain a surprising connection between the competitive equilibria of supply-aware markets, and symmetric NE of the exchange market game, and as a consequence get both the welfare theorems [32] for supply-aware markets. In addition we get connectedness of the CE set, and a characterization for the uniqueness of CE, for the case of linear utilities.

1.1 Brief overview of main results

We extend the game analyzed in [1,7,8] for Fisher markets, to the setting of exchange markets. To start with we consider markets where agents have linear utility functions, also called *linear markets*. In an exchange market $\mathcal{M}$, an agent's utility function is private to her and hence *strategizable*, while she must disclose her initial endowment of goods in order to sell and hence is non-strategizable. In an *exchange market game* $\Gamma(\mathcal{M})$, agents report (play) linear utility functions, and the game calculates a competitive equilibrium (CE) prices and allocations based on the reported utilities, and distributes the goods accordingly (see Section 2 for details). However the issue is: among many different competitive equilibria of the played market, which one to chose to decide the outcome. For linear exchange markets the set of competitive equilibrium prices is convex, and the set of all equilibrium allocations remains the same for every equilibrium price [16,17]. Thus regardless of what prices we chose, there is an obvious choice for the outcome allocation: the one maximizing social welfare of agents.

We say that a strategy profile is *conflict-free*, if there exists an allocation preferred by all the agents, among all the equilibrium allocations of the played market. Clearly if there is such an allocation, then it will be chosen as the outcome. We analyze the symmetric strategy profiles of exchange market game, where all agents play the same strategy, and show the following (see Section 3).

Theorem 1 (Informal).

- *A symmetric strategy profile is a Nash equilibrium if and only if it is conflict-free.*
- *The payoffs achieved at symmetric Nash equilibria are Pareto-optimal.*
- *The symmetric Nash equilibria form a connected set, and there exists necessary and sufficient condition for its uniqueness.*
- *Every competitive equilibrium prices of the true market (with true utility functions) can be achieved at one of its symmetric Nash equilibria.*

For exchange markets with arbitrary concave utility functions, we show the following, where played utility functions are still restricted to be linear.

Theorem 2 (Informal).

- *If a symmetric strategy profile is conflict-free then it gives a Nash equilibrium, and the payoffs at such Nash equilibria are Pareto-optimal.*

– *Every competitive equilibrium prices of the true market can be achieved at a symmetric Nash equilibrium.*

Next we analyze the market where agents do not strategize, but are aware of the total supply of goods. Recall that in the exchange market, every agent demands a (optimal) utility maximizing bundle that is affordable at any given prices, and if the market clears after every agent gets her bundle then the prices give a CE. For the case where agents are aware of the supply of goods, they will calculate their optimal bundles accordingly at given prices, *i.e.,* even if a good is most preferred at these prices to an agent, her demand for the good does not exceed the supply. This changes the dynamics of demand bundles a great deal, and the question is how, as a consequence, the competitive equilibrium points will change. We call such a market as *supply aware exchange market*, denote it by $\mathcal{M}^{SA}$, and show the following (see Section 4 for details).

Theorem 3 (Informal).

– *Prices and allocation give a competitive equilibrium of a supply-aware linear exchange market* $\mathcal{M}^{SA}$ *if and only if they can be achieved at a symmetric Nash equilibrium of the game* $\Gamma(\mathcal{M})$.
– *Competitive equilibrium prices and allocation, of a supply-aware exchange market with arbitrary concave utility functions, can be achieved at a symmetric Nash equilibria of the game* $\Gamma(\mathcal{M})$.

As corollaries of Theorems 1, 2 and 3, we get the first and second welfare theorems [32] for the supply-aware exchange markets. Further, for linear supply-aware markets we get that its CE set is connected but not convex, and the characterization for its uniqueness.

The computation of a competitive equilibrium of an exchange market with separable piecewise linear concave (PLC) function is PPAD-hard, even when the PLC function for each agent-good pair has exactly two segments, with zero slope for the second segment [9]. Further, the set of competitive equilibria prices of these markets can be disconnected. We note that, supply-aware exchange market with linear utilities is a special case of this market where the second segment starts at an amount equal to the total supply of the respective good in each function. This restriction surprisingly makes the market well behaved, in the sense that the set of CE prices is connected, and equilibrium computation is efficient as it suffices to find a CE of linear exchange market [20].

1.2 Related Work

Shapley and Shubik [28] consider a market game for the exchange economy, where every good has a trading post, and the strategy of a buyer is to bid (money) at each trading post. For each strategy profile, the prices are determined naturally so that market clears and goods are allocated accordingly, however agents may not get their optimal bundles. Many variants [2,13] of this game have been extensively studied. Essentially, the goal is to design a mechanism to implement competitive equilibrium (CE), *i.e.*, to capture CE at a NE of the game. The strategy space of this game is tied to the implementation of the market (in this case, trading posts). Our strategy space is the utility functions itself, and is independent of the market implementation.

In word auction markets as well, a similar study on strategic behavior of buyers (advertisers) has been done [14,32].

In next few pages, we present main ideas, techniques and results of the paper. Due to space constrains, some of the proofs are omitted from the main paper, and can be found in Appendix A, unless specified otherwise.

2 Exchange Market Game

In this section first we briefly describe the exchange market model [33] and later define a game on these market, that is an extension of the game defined in [1].

The exchange market consists of a set $\mathcal{G}$ of goods and a set $\mathcal{A}$ of agents. Let n denote the number of goods and m denote the number of agents in the market. Each agent has an initial endowment of goods, for agent i it is $\boldsymbol{w}_i = (w_{i1}, \ldots, w_{in})$ where w_{ij} is the amount of good j with agent i. Further, she wants to buy a (optimal) bundle of goods that maximizes her utility to the extent allowed by the money earned by selling her initial endowment. The preference of an agent i over bundles of goods can be captured by a non-negative, non-decreasing and concave utility function $U_i : \mathbf{R}^n_+ \rightarrow \mathbf{R}_+$. Non-decreasingness models free disposal property, and concavity models the law of diminishing marginal returns. Without loss of generality (wlog) we assume that total available quantity of each good is one[5], *i.e.,* $\sum_{i\in\mathcal{A}} w_{ij} = 1, \forall j \in \mathcal{G}$. We denote this market by $\mathcal{M}$.

Given prices $\boldsymbol{p} = (p_1, \ldots, p_n)$, agent i earns $\boldsymbol{w}_i \cdot \boldsymbol{p}$ by selling her initial endowment, and demands an affordable bundle $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{in})$ maximizing her utility (optimal bundle). Prices $\boldsymbol{p}$ are said to give a *competitive equilibrium* (CE) if, there is an assignment of an optimal bundle to every agent, and demand equals supply, *i.e.,* market clears.

We are going to consider markets where utility functions are linear; *linear markets.* For an agent i, her utility function U_i is defined as $U_i(\boldsymbol{x}_i) = \sum_{j\in\mathcal{G}} u_{ij}x_{ij}$; agent i gets u_{ij} amount of utility from a unit amount of good j. We assume that every good is liked by some agent, *i.e.,* $\forall j,\ u_{ij} > 0$ for some i, or else it can be distributed freely. Let $\boldsymbol{u}_i$ denote the vector $(u_{i1}, \ldots, u_{in})$. The following conditions are necessary and sufficient for $\boldsymbol{p}$ and $\boldsymbol{x}$ to be a CE allocation and prices [33].

$$\forall i \in \mathcal{A}, \forall j \in \mathcal{G} : \quad x_{ij} > 0 \Rightarrow \tfrac{u_{ij}}{p_j} \geq \tfrac{u_{ij'}}{p_{j'}}, \quad \forall j' \in \mathcal{G} \tag{1}$$

$$\begin{array}{ll} \forall j \in \mathcal{G} : & \sum_{i\in\mathcal{A}} x_{ij} = 1 \\ \forall i \in \mathcal{A} : & \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{w}_i \cdot \boldsymbol{p} \end{array} \tag{2}$$

[5] This is like redefining the unit of goods by appropriately scaling utility parameters.

In (1) the ratio ${}^{u_{ij}}/_{p_j}$ is the marginal utility per unit money of agent i for good j at prices $\boldsymbol{p}$, and hence she wants to buy only those which maximize this ratio. The last two conditions ensure market clearing. Note that if $\boldsymbol{p}$ is a CE price vector, then so is $\alpha\boldsymbol{p}$, $\forall\alpha > 0$. Henceforth, wlog we assume that $\sum_{j\in\mathcal{G}} p_j = 1$.[6] Also, for an agent i since scaling u_{ij}'s by a positive constant does not change the CE, we assume that $\sum_{j\in\mathcal{G}} u_{ij} = 1,\ \forall i \in \mathcal{A}$.

It is known that exchange markets are not incentive compatible [1,7,8,27][7], and agents may gain by reporting fictitious utility functions. The following example illustrates the same.

Example 1. Consider a market with two goods, and two agents with linear utility functions. Let $\boldsymbol{w}_1 = \boldsymbol{w}_2 = (0.5, 0.5)$, $U_1(x_1, x_2) = 2x_1 + x_2$ and $U_2(x_1, x_2) = x_1 + 2x_2$. Equilibrium prices are $(p_1, p_2) = (1, 1)$ and allocations are $\boldsymbol{x}_1 = (1, 0)$ and $\boldsymbol{x}_2 = (0, 1)$. Now, if agent 1 poses her utility function as $U_1' = U_2$ instead, then the equilibrium prices will be $(p_1', p_2') = (1, 2)$ and allocations will be $\boldsymbol{x}_1' = (1, 0.05)$ and $\boldsymbol{x}_2' = (0, 0.95)$. Since $\boldsymbol{x}_1 < \boldsymbol{x}_1'$ agent 1 gains by deviating.

Based on the observation of Example 1 next we define exchange market game; an extension of game defined in [1] for Fisher markets (a special case of exchange markets). Given a linear exchange market $\mathcal{M}$ consider a single-shot, non-cooperative exchange market game $\Gamma(\mathcal{M})$, where agents are the players. We assume that the agents' endowments are common knowledge, while the utility functions are their private information and hence strategizable. This is because, the endowments, when put up on sell, become public knowledge, while utility function of an agent is still known only to her. In a play, agents report their utility functions, and each receives a bundle of goods as per a competitive equilibrium of the market with reported utilities. We will call the market $\mathcal{M}$ as the *true market*, the market in a play of game $\Gamma(\mathcal{M})$ with possibly fictitious utility functions as *played market*.

The set of strategies for an agent is the set of all linear utility functions from $\mathbb{R}^n$ to $\mathbb{R}$, up to scaling. Therefore, the strategy set S_i of agent i is Δ_n, where Δ_n denotes the n-dimensional simplex. The game is played as follows: Suppose agent i reports $\boldsymbol{s}_i \in S_i$. First we compute competitive equilibrium allocation for the played market. It is known that the set of competitive equilibrium allocations forms a convex set in case of linear exchange markets [16]. For a strategy profile $\boldsymbol{s} \in S = \times_{i\in\mathcal{A}} S_i$, we denote this set by $\mathcal{X}(\boldsymbol{s})$. Outcome of a play is an allocation $\boldsymbol{x}(\boldsymbol{s})$ achieving maximum social welfare as per the true utility functions, and also balanced payoffs whenever there is a choice.

$$\boldsymbol{x}(\boldsymbol{s}) = \arg\max_{\boldsymbol{x}\in\mathcal{X}(\boldsymbol{s})} \prod_i U_i(\boldsymbol{x}_i) \tag{3}$$

Since, product is a strictly concave function, there will be a unique allocation achieving the maximum. One can think of the outcome process as, once prices

[6] We may not follow this assumption in examples to work with simpler numbers.

[7] These results are for Fisher markets which is a special case of exchange markets.

are set based on the reported utility functions, every agent will try to buy the best possible bundle at the given prices, and hence socially optimal allocation is reached.

Remark 1. There may not exist a CE in an exchange market [3,23]. If such a case happens for the played market, then the outcome of the play could be: agents keep their own endowments. Assuming $w_{ij} > 0, \forall (i,j)$ will avoid such a case [3].

3 Nash Equilibrium Characterization

In this section we derive necessary and sufficiency conditions for the symmetric Nash equilibria (NEs) of game $\Gamma(\mathcal{M})$, and using this characterization we derive important properties like, Pareto-optimality, achieving CE prices at a symmetric NE (SNE), connectedness of the SNE set, and necessary and sufficient conditions for the uniqueness of the SNE prices.

Note that, in a played market with strategy profile $\boldsymbol{s}$, marginal utility per unit of money of agent i for good j at prices $\boldsymbol{p}$ is ${s_{ij}}/{p_j}$ (due to (1)). Therefore, at prices $\boldsymbol{p}$, agent i can be assigned only those goods which have this ratio as $\max_l {s_{il}}/{p_l}$, and any amounts of them. Let $G(\boldsymbol{s}, \boldsymbol{p})$ be the bipartite graph between agents and nodes, with an edge between agent i and good j if $j \in \max_l {s_{il}}/{p_l}$, *i.e.,* if good j can be assigned to agent i at prices $\boldsymbol{p}$ when $\boldsymbol{s}$ is played.

A strategy profile $\boldsymbol{s}$ is said to be symmetric if all the players play the same strategy in $\boldsymbol{s}$, *i.e.,* $\boldsymbol{s}_i = \boldsymbol{s}_{i'}, \forall i, i' \in \mathcal{A}$.

Lemma 1. *If a strategy profile $\boldsymbol{s}$ is symmetric, then the only CE prices of played market with utilities $\boldsymbol{s}$ and endowment $\boldsymbol{w}_i$'s are $\boldsymbol{p} = \boldsymbol{s}_i$.*

Proof. At $\boldsymbol{p} = \boldsymbol{s}_i$ the ratio ${s_{ij}}/{p_j}$ is one for all (i,j). Clearly, $G(\boldsymbol{s}, \boldsymbol{p})$ is a complete bipartite graph. Therefore, there exists a market clearing assignment and hence $\boldsymbol{p}$ is a CE. Further, if there is any other CE prices $\boldsymbol{p}' \neq \boldsymbol{p}$, then $\exists j \neq j'$ such that ${s_{ij}}/{p'_j} < {s_{ij}}/{p'_{j'}}, \forall i \in \mathcal{A}$. In that case, no agent will want to buy good j at prices $\boldsymbol{p}'$ (see (1)), and hence market for good j will not clear. $\square$

Let $\pi_i(\boldsymbol{s})$ be the maximum payoff that agent i can achieve from any allocation of $\mathcal{X}(\boldsymbol{s})$, *i.e.,* $\pi_i(\boldsymbol{s}) = \max_{\boldsymbol{x} \in \mathcal{X}(\boldsymbol{s})} U_i(\boldsymbol{x}_i)$.

Definition 1. *A strategy profile $\boldsymbol{s} \in S$ is said to be* conflict-free *if there exists $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{s})$ such that $U_i(\boldsymbol{x}_i) = \pi_i(\boldsymbol{s})$, $\forall i \in \mathcal{A}$. Such an allocation is called a* conflict-free *allocation of $\boldsymbol{s}$.*

Note that, if a played strategy profile $\boldsymbol{s}$ has a conflict-free allocation in $\mathcal{X}(\boldsymbol{s})$ then clearly that will be chosen as the outcome allocation $\boldsymbol{x}(\boldsymbol{s})$ by (3), and every agent gets the best possible payoff $\pi_i(\boldsymbol{s})$. It turns out that similar to [1], the notion of *conflict-free* utilities is pivotal in characterizing Nash equilibria even for such a general setting of exchange market game with arbitrary concave utilities.

Lemma 1 implies that if $\boldsymbol{s}$ is symmetric then the played market has a unique equilibrium prices namely $\boldsymbol{p} = \boldsymbol{s}_i$. Clearly, $G(\boldsymbol{s}, \boldsymbol{p} = \boldsymbol{s}_i)$ is a complete bipartite graph in that case. Due to this, any bundle $\boldsymbol{x}_k$ that agent k can afford at prices $\boldsymbol{p}$, is feasible at $\boldsymbol{s}$, *i.e.,* $\boldsymbol{x}_k \cdot \boldsymbol{p} = \boldsymbol{w}_k \cdot \boldsymbol{p} \Rightarrow \exists \boldsymbol{x}' \in \mathcal{X}(\boldsymbol{s}),\ \boldsymbol{x}_k = \boldsymbol{x}'_k$. We use these facts crucially in the proofs that follows.

Lemma 2. *A symmetric Nash equilibrium strategy profile has to be conflict-free.*

Proof. Suppose $\boldsymbol{s}$ is a symmetric Nash equilibrium, but not conflict-free. Let $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{s})$ be the outcome allocation for the play $\boldsymbol{s}$. Then, there exists an agent k with $U_k(\boldsymbol{x}_k) < \pi_k(\boldsymbol{s})$. Let $\boldsymbol{x}' \in \mathcal{X}(\boldsymbol{s})$ be the allocation achieving the maximum payoff for agent k, *i.e.,* $U_i(\boldsymbol{x}'_k) = \pi_k(\boldsymbol{s})$. Let $\boldsymbol{p}$ be an equilibrium prices for the played market. Since graph $G(\boldsymbol{s}, \boldsymbol{p})$ is a complete bipartite graph (Lemma 1), there are cycles involving agent k, and therefore many allocations are possible. We will break all these cycles by perturbing $\boldsymbol{s}_k$ so that the only feasible allocation for agent k is a perturbed version of $\boldsymbol{x}'_k$.

Let $\boldsymbol{x}'$ be such that agent k is sharing at most one good with any other agent. Such an allocation exist because $G(\boldsymbol{s}, \boldsymbol{p})$ is a complete bipartite graph, and at prices $\boldsymbol{p}$ agent k will want to buy goods in the decreasing order of $^{u_{kj}}/_{p_j}$, and she is indifferent between goods having the same value for it. Let $J_1 = \{j \in \mathcal{G} \mid x'_{kj} = 1\}$, $J_2 = \{j \in \mathcal{G} \mid x'_{kj} = 0\}$, and let j' be the shared good.

Consider a deviating strategy $\boldsymbol{s}'_k$ for agent k, where $s'_{kj} = p_j + \epsilon,\ \forall j \in J_1$, $s'_{kj'} = p_{j'}$, and $s'_{kj} = 0, \forall j \in J_2$, with a small positive constant ϵ. Rescale s'_{kj}s so that they sum up to one. Let $\boldsymbol{s}' = (\boldsymbol{s}'_k, \boldsymbol{s}_{-k})$ be the new strategy profile.

Set p'_j to $p_j + \epsilon$ if $j \in J_1$ and p_j otherwise, and scale them so that they sum to one. Clearly, in graph $G(\boldsymbol{s}', \boldsymbol{p}')$ only agent k has edges to goods of J_1, all the agents have edges to good j', and all except k have edges to goods of J_3. Let $\delta = \pi_k(\boldsymbol{s}) - U_k(\boldsymbol{x}_k)$. If we set ϵ to less than $\min\{^{\delta p_{j'}}/_{(u_{kj'}|J_1|(2+|J_1|))},\ \min_{x'_{ij}>0} {}^{x'_{ij}}/_{m*n}\}$, then it is easy to show that $\boldsymbol{p}'$ is the CE prices of the played market for profile $\boldsymbol{s}'$. Further, in every allocation of $\mathcal{X}(\boldsymbol{s}')$ all the goods of J_1 goes to agent k, and any $\boldsymbol{x}' \in \mathcal{X}(\boldsymbol{s}')$ is strictly better than the outcome of play $\boldsymbol{s}$ for agent k, *i.e.,* $U_k(\boldsymbol{x}'_k) > U_k(\boldsymbol{x}_k)$. □

Next we show that conflict-freeness is enough for a symmetric strategy to be a Nash equilibrium, and thus we get a complete characterization of symmetric NEs of game $\Gamma(\mathcal{M})$.

Lemma 3. *If a symmetric strategy profile $\boldsymbol{s}$ is conflict-free then it gives a Nash equilibrium of game $\Gamma(\mathcal{M})$.*

Proof. To the contrary suppose not. Then $\exists k \in \mathcal{A}$ who can deviate to $\boldsymbol{s}'_k$ and gain. Let $\boldsymbol{s}' = (\boldsymbol{s}'_k, \boldsymbol{s}_{-k})$ be the strategy profile when k deviates, and let $\boldsymbol{x}' = \boldsymbol{x}(\boldsymbol{s}')$. We will construct an allocation $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{s})$ such that $\boldsymbol{x}_k \geq \boldsymbol{x}'_k$. This will prove the lemma, since payoff of agent k while playing $\boldsymbol{s}_k$ is $\pi_k(\boldsymbol{s}) \geq U_k(\boldsymbol{x}) \geq U_k(\boldsymbol{x}'_k)$; the last inequality is due to non-decreasingness of U_k.

Due to Lemma 1 the equilibrium prices for $\boldsymbol{s}$ is unique, denote them by $\boldsymbol{p}$. Suppose, the CE prices are unique for profile $\boldsymbol{s}'$ as well and let they be $\boldsymbol{p}'$. Let

$J_1 = \{j \in \mathcal{G} \mid {p'_j}/{p_j} = \min_l {p'_l}/{p_l}\}$, and $J_2 = \mathcal{G} \setminus J_1$. Clearly, all the agents except k will want to buy only goods of J_1 at $\boldsymbol{p}'$, hence k has to buy all of J_2 in $\boldsymbol{x}'$. Let $a_1 = \sum_{j \in J_1} w_{kj} p'_j$ and $a_2 = \sum_{j \in J_2} w_{kj} p'_j$. Market clearing condition at $\boldsymbol{p}'$ gives,

$$\begin{aligned} \boldsymbol{w}_k \cdot \boldsymbol{p}' = \boldsymbol{x}'_k \cdot \boldsymbol{p} &\Rightarrow a_1 + a_2 = \textstyle\sum_{j \in J_1} x'_{kj} p'_j + \sum_{j \in J_2} p'_j \\ &\Rightarrow a_1 = \textstyle\sum_{j \in J_1} x'_{kj} p'_j + \sum_{j \in J_2} p'_j - \sum_{j \in J_2} w_{kj} p'_j \\ &\Rightarrow a_1 = \textstyle\sum_{j \in J_1} x'_{kj} p'_j + \sum_{j \in J_2} (1 - w_{kj}) p'_j \end{aligned}$$

Let $\alpha = {p_j}/{p'_j},\ j \in J_1$ and $\forall j \in J_2,\ \beta_j = {p_j}/{p'_j}$. By construction we have $\alpha > \beta_j, \forall j \in J_2$. Continuing with the above derivation,

$$\begin{aligned} &\Rightarrow \alpha a_1 = \textstyle\sum_{j \in J_1} x'_{kj} (\alpha p'_j) + \sum_{j \in J_2} (1 - w_{kj})(\alpha p'_j) \\ &\Rightarrow \textstyle\sum_{j \in J_1} w_{kj} (\alpha p'_j) \geq \sum_{j \in J_1} x'_{kj} p_j + \sum_{j \in J_2} (1 - w_{kj}) p_j \\ &\Rightarrow \textstyle\sum_{j \in \mathcal{G}} w_{kj} p_j \geq \sum_{j \in J_1} x'_{kj} p_j + \sum_{j \in J_2} p_j \\ &\Rightarrow \boldsymbol{w}_k \cdot \boldsymbol{p} \geq \boldsymbol{x}'_k \cdot \boldsymbol{p} \end{aligned}$$

The above expression implies that at prices $\boldsymbol{p}$, agent k earns at least as much as the money required to buy bundle $\boldsymbol{x}'_k$. Furthermore, while going from $\boldsymbol{p}'$ to $\boldsymbol{p}$ the earnings of all other agents will scale at most by α. Therefore, they can barely afford the goods of J_1 that they are getting in $\boldsymbol{x}'$, at prices $\boldsymbol{p}$. Since, $\boldsymbol{p} = \boldsymbol{s}_i, \forall i \in \mathcal{A}$ (Lemma 1), $G(\boldsymbol{s}, \boldsymbol{p})$ is a complete bipartite graph. Therefore, $\exists \boldsymbol{x} \in \mathcal{X}(\boldsymbol{s})$ such that $\boldsymbol{x}_k \geq \boldsymbol{x}'_k$.

In case of linear utilities, the set of optimal allocation for every equilibrium prices remains the same [16]. Hence, even if the equilibrium prices are not unique for $\boldsymbol{s}'$, considering any of them will work. □

The next theorem completely characterizes the symmetric Nash equilibria, and follows directly using Lemma 2 and Lemma 3

Theorem 4. *A symmetric strategy profile gives a Nash equilibrium of game $\Gamma(\mathcal{M})$ iff it is conflict-free.*

Using the characterization of Theorem 4 we derive a number of properties of the symmetric Nash equilibria. The crucial question in any strategic analysis of markets is whether a competitive equilibrium allocation, which is assumed to be efficient, can be achieved at any Nash equilibrium [28,2,13]. We show that, for game $\Gamma(\mathcal{M})$, the answer to this question is yes for all possible competitive equilibrium allocations.

Lemma 4. *Every CE prices and allocation of the true market $\mathcal{M}$ can be achieved at a symmetric NE of game $\Gamma(\mathcal{M})$.*

The first fundamental theorem of welfare economics states that the payoff achieved at any competitive equilibrium is Pareto-optimal [32]. The next theorem establishes similar result for the symmetric Nash equilibria. Let $\mathcal{U}$ be the set of all possible utility-tuples achievable by any feasible allocation, *i.e.,*

$$\mathcal{U} = \{(U_1(\boldsymbol{x}_1), \ldots, U_m(\boldsymbol{x}_m)) \mid \sum_{i \in \mathcal{A}} x_{ij} \leq 1,\ \forall j \in \mathcal{G}\} \tag{4}$$

Lemma 5. *Let $\boldsymbol{s}'$ be a symmetric Nash equilibrium with $\boldsymbol{x}' = \boldsymbol{x}(\boldsymbol{s}')$, then $(U_1(\boldsymbol{x}'_1), \dots, U_m(\boldsymbol{x}'_m))$ is a Pareto-optimal point of $\mathcal{U}$.*

General exchange market can have multiple competitive equilibria, and they may form a disconnected set [11]. The uniqueness of equilibrium, competitive or Nash, is a very important property, and a lot of work has been done to characterize such instances [26,24,25,21,12]. The following lemma derives such a characterization for the uniqueness of symmetric Nash equilibria in game $\Gamma(\mathcal{M})$.

Lemma 6. *The game $\Gamma(\mathcal{M})$ has a unique symmetric Nash equilibrium iff market $\mathcal{M}$ has a unique CE prices, say $\boldsymbol{p}^*$, and in graph $G(\boldsymbol{u}, \boldsymbol{p}^*)$ degree of every good is at least two.*

In case of linear exchange markets, the competitive equilibrium prices form a convex set [16], and the proof is quite involved. Next we show that the set of symmetric Nash equilibria of game $\Gamma(\mathcal{M})$ forms a connected set, and again the proof is quite involved, discussed in Appendix B.

Lemma 7. *The set of symmetric Nash equilibrium prices form a connected set.*

3.1 Exchange market game with concave utility functions

We extend results of the previous section to general exchange markets where the utility functions of the agents are arbitrary concave, non-decreasing functions. Let $\mathcal{M}$ be such an exchange market, and consider a game $\Gamma(\mathcal{M})$ where strategies of the agents are still restricted to linear functions. Therefore, the strategy sets of agents, the outcome function of the game, namely (3), and the notion of conflict-free strategies remain unchanged.

First we show that conflict-freeness is sufficient for a symmetric Nash equilibrium in such a general setting as well.

Lemma 8. *If a symmetric strategy profile is conflict-free, then it is a Nash equilibrium of game $\Gamma(\mathcal{M})$.*

Proof. Since, played utilities are still linear, the proof is essentially same as the proof of Lemma 3. Because it is all about constructing an allocation where the deviating agent gets at least as much as what she gets when she deviates. $\square$

An exchange market $\mathcal{M}$ may have many disconnected competitive equilibria [11]. Using the above lemma next we show that all of these can be achieved at symmetric NE of game $\Gamma(\mathcal{M})$, a desirable property for any market game.

Lemma 9. *A competitive equilibrium prices and allocation of the true market $\mathcal{M}$ can be achieved at a symmetric NE of game $\Gamma(\mathcal{M})$.*

Let $\mathcal{U}$ be the set of all possible payoff tuples, as defined by (4). We show that for this general setting too, allocations at symmetric Nash equilibria are efficient, in the sense that they achieve Pareto-optimal payoffs.

Lemma 10. *The payoffs achieved at a symmetric NE of game $\Gamma(\mathcal{M})$ with conflict-free allocation, gives a Pareto-optimal point of set $\mathcal{U}$.*

4 Supply Aware Exchange Market

Supply aware exchange markets are similar to exchange markets, except for one crucial difference. In exchange markets agents are assumed to be unaware of the total supply of goods. Therefore, an agent does not take the supply in to account when she calculates her optimal bundle at the given prices $\boldsymbol{p} = (p_1, \ldots, p_n)$. This need not be the case always, and agents may take supply into considerations as well. Suppose, agents are aware of the total supply of goods in the market; 1 unit of each good. Then, at the given prices $\boldsymbol{p}$, agent i will solve the following problem instead, to calculate her optimal bundle.

$$\begin{aligned} &\max : U_i(\boldsymbol{x}_i) \\ &s.t. \quad \boldsymbol{x}_i \cdot \boldsymbol{p} \le \boldsymbol{w}_i \cdot \boldsymbol{p} \\ &\qquad\ \ 0 \le x_{ij} \le 1, \quad \forall j \in \mathcal{G} \end{aligned} \tag{5}$$

We will call such a market where agents are aware of supply of every good, as *Supply Aware Exchange Markets* (SAEM), and denote it by $\mathcal{M}^{SA}$. Again prices $\boldsymbol{p}$ are said to be competitive equilibrium of market $\mathcal{M}^{SA}$ if, every agent gets an optimal bundle and there is no surplus or deficiency of any good, *i.e.,* market clears. Let $\boldsymbol{p}$ be a CE prices and $\boldsymbol{x}$ be its optimal allocation, then the market clearing condition can be formally stated as,

$$\begin{aligned} &\forall j \in \mathcal{G}, \quad \textstyle\sum_{i \in \mathcal{A}} x_{ij} = 1 \\ &\forall i \in \mathcal{A}, \quad\ \ \boldsymbol{x}_i \cdot \boldsymbol{p} \le \boldsymbol{w}_i \cdot \boldsymbol{p} \end{aligned} \tag{6}$$

Competitive equilibrium in an exchange market exists under mild conditions [3,23]. An immediate question is, whether they always exist in supply-aware exchange markets too. In the next lemma we show that a supply-aware market admits a CE whenever corresponding exchange market has one.

Lemma 11. *Every competitive equilibrium of the exchange market $\mathcal{M}$ is also an equilibrium of the corresponding supply aware market $\mathcal{M}^{SA}$.*

Proof. Let $\boldsymbol{p}^*$ be a CE prices of the exchange market $\mathcal{M}$. Agent i solves the following program to calculate her optimal bundle in $\mathcal{M}$.

$$\begin{aligned} &\max : U_i(\boldsymbol{x}_i) \\ &s.t. \quad \boldsymbol{x}_i \cdot \boldsymbol{p} \le \boldsymbol{w}_i \cdot \boldsymbol{p} \\ &\qquad\ \ x_{ij} \ge 0, \quad \forall j \in \mathcal{G} \end{aligned}$$

Let $\boldsymbol{x}^*$ be the optimal allocation at prices $\boldsymbol{p}^*$, then $\boldsymbol{x}_i^*$ is an optimal solution of the above program, and $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies (6) too to ensure market clearing. Hence, $x_{ij}^* \le 1, \forall (i,j)$. This implies that $\boldsymbol{x}_i^*$ is an optimal solution of program (5) too at prices $\boldsymbol{p}^*$. Thus $\boldsymbol{x}_i^*$ is a optimal bundle of agent i at prices $\boldsymbol{p}^*$ in the supply-aware market $\mathcal{M}^{SA}$ too, and hence prices $\boldsymbol{p}^*$ and allocation $\boldsymbol{x}^*$ gives a CE of the supply aware market $\mathcal{M}^{SA}$ as well. $\square$

Existence of an equilibrium in a supply aware market follows from the above lemma and the Arrow-Debreu theorem on existence of CE in exchange markets

[3]. However, there may be many more equilibria in market $\mathcal{M}^{SA}$ as demonstrated by the following example. In all the examples that follow, we consider a market with two goods, and two agents with linear utility functions.

Example 2. Let $\boldsymbol{w}_1 = \boldsymbol{w}_2 = (0.5, 0.5)$ be the endowments, and $U_1(x_1, x_2) = 2x_1 + x_2$ and $U_2(x_1, x_2) = x_1 + 2x_2$ be the utility functions. The only equilibrium prices of the corresponding exchange market $\mathcal{M}$ is $(1, 1)$. However, every convex combination of points $(2, 1)$ and $(1, 2)$ gives an equilibrium prices for the supply aware market $\mathcal{M}^{SA}$. $\square$

It is also possible that a market $\mathcal{M}$ does not have a CE[8] but corresponding supply aware market $\mathcal{M}^{SA}$ has one, as illustrated by the following example.

Example 3. Consider a market with two goods, and two agents with linear utility functions. Let $\boldsymbol{w}_1 = (1, 0.5)$ and $\boldsymbol{w}_2 = (0, 0.5)$ be the endowment vectors. Let the utility functions be $U_1(x_1, x_2) = x_1$ and $U_2(x_1, x_2) = x_2 + {}^{x_1}/_{2}$.

In case of exchange market setting $\mathcal{M}$ if $p_2 > 0$ then agent 1 will have more money than what she can spend and hence market will not clear. If p_2 is set to zero then agent 2 will ask for an infinite amount of good 2 as she is unaware of the supply. Therefore, there does not exist an equilibrium in market $\mathcal{M}$.

However, in market $\mathcal{M}^{SA}$ where agents are aware of the supply, setting prices to $(p_1, p_2) = (1, 0)$ gives an equilibrium. $\square$

A market is said to satisfy *weak gross substitute* (WGS) property, if increase in price of good j does not decrease the demand of any other good [32]. Exchange markets with linear utilities is one such example. The next example demonstrates that even if an exchange market $\mathcal{M}$ satisfies WGS property, the corresponding supply-aware market need not be WGS.

Example 4. Consider an exchange market $\mathcal{M}$ with linear utilities, with two goods and two agents. Let $\boldsymbol{w}_1 = (0.8, 0.6)$, $\boldsymbol{w}_2 = (0.2, 0.4)$, $U_1(x_1, x_2) = 10x_1 + x_2$ and $U_2(x_1, x_2) = 5x_1 + x_2$. At prices $(p_1, p_2) = (1, 1)$, the optimal bundles of agent 1 and 2 are $(1, 0.4)$ and $(0.6, 0)$. Thus demands of good 1 and 2 are 1.6 and 0.4 respectively. If we increase p_1 to 2, then optimal bundles are $(1, 0.2)$ and $(0.4, 0)$, and hence demand of good 2 decreases to 0.2. $\square$

5 Nash meets Walras

In this section we establish an equivalence between the competitive equilibria of supply-aware market $\mathcal{M}^{SA}$ and the symmetric Nash equilibria of exchange market game $\Gamma(\mathcal{M})$, for the linear case. As consequences of this equivalence, we derive a number of properties for the supply-aware markets, like both the welfare theorems [32], and connectedness of the CE set, and characterization for the uniqueness of CE. For markets with arbitrary concave utilities we show that

[8] The weakest known sufficiency conditions for existence of a competitive equilibrium in exchange markets is given by Maxfield [23]

the former is a subset of the latter, which allows us to derive both the welfare theorems for this general setting as well.

We start with the general setting first. Given a supply-aware exchange market $\mathcal{M}^{SA}$ with concave utility functions, we show that its equilibrium can be achieved at a symmetric NE of game $\Gamma(\mathcal{M})$.

Lemma 12. *Every equilibrium prices and allocations of market $\mathcal{M}^{SA}$ can be achieved at a symmetric Nash equilibrium of game $\Gamma(\mathcal{M})$.*

Proof. Let $\boldsymbol{p}^*$ and $\boldsymbol{x}^*$ be a CE prices and allocation of market $\mathcal{M}^{SA}$. Set the strategy profile $\boldsymbol{s}^*$ be such that $\boldsymbol{s}_i^* = \boldsymbol{p}$. Clearly, $\boldsymbol{x}^* \in \mathcal{X}(\boldsymbol{s}^*)$, since (6) is satisfied and all the agents like all the goods as per utilities $\boldsymbol{s}^*$ and prices $\boldsymbol{p}^*$. Since every agent gets an optimal bundle in $\boldsymbol{x}^*$ at prices $\boldsymbol{p}^*$, it is also a conflict-free allocation of $\boldsymbol{s}^*$. Thus, due to Theorem 4 for the linear case, and Lemma 8 for the arbitrary concave case, strategy profile $\boldsymbol{s}^*$ is a symmetric NE achieving prices $\boldsymbol{p}^*$. $\square$

Using the above lemma and results established in Section 3, we show that (as expected) the two fundamental theorem of welfare economics [32] follow for the supply-aware markets as well. The next theorem is a direct consequence of Lemmas 12 and 10.

Theorem 5 (First Welfare Theorem). *The payoffs achieved at a competitive equilibrium of market $\mathcal{M}^{SA}$ are Pareto-optimal.*

The second welfare theorem states that given utility functions U_i's, every Pareto-optimal points of $\mathcal{U}$ of (4) can be achieved at a CE of some exchange market with U_i utility functions and some endowment vectors. This theorem trivially follows for supply aware markets using Lemmas 12 and 9 (also Lemma 11 suffices), together with the fact that the theorem holds for the exchange markets [32].

Theorem 6 (Second Welfare Theorem). *Every Pareto-optimal point of $\mathcal{U}$ can be achieved at a CE of market $\mathcal{M}^{SA}$ with utility functions U_i and some endowment vectors.*

Going back to the connection between competitive equilibria of supply-aware and symmetric NE of the game, for linear markets we get containment in other direction as well, as shown in the next lemma.

Lemma 13. *Every symmetric Nash equilibrium prices and allocations of game $\Gamma(\mathcal{M})$ can be achieved at an equilibrium of market $\mathcal{M}^{SA}$, when utility function U_i's are linear.*

Proof. Let $\boldsymbol{s}^*$ be a symmetric NE of game $\Gamma(\mathcal{M})$ and $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{s}^*)$ be its conflict-free allocation (Theorem 4). Prices at $\boldsymbol{s}^*$ is $\boldsymbol{p}^* = \boldsymbol{s}_i^*$ (Lemma 1). Clearly, $\exists \boldsymbol{x}' \in \mathcal{X}(\boldsymbol{s}^*)$ such that $\boldsymbol{x}_i'$ is an optimal bundle of agent i at prices $\boldsymbol{p}^*$ in market $\mathcal{M}^{SA}$. Since, $\boldsymbol{x}^*$ is conflict-free, we have $U_i(\boldsymbol{x}_i^*) \geq U_i(\boldsymbol{x}_i')$. Therefore, $\boldsymbol{x}_i^*$ is also an optimal bundle of agent i. This is true for all the agents. Therefore, at prices $\boldsymbol{p}^*$, allocation $\boldsymbol{x}^*$ is optimal and market clearing, hence is a CE of $\mathcal{M}^{SA}$. $\square$

The next theorem follows using Lemmas 12 and 13.

Theorem 7. *The set of equilibrium prices of a linear market $\mathcal{M}^{SA}$ is exactly the set of prices achieved at symmetric Nash equilibria of game $\Gamma(\mathcal{M})$.*

Since, the exchange market game $\Gamma(\mathcal{M})$ exhibits nice structural properties, for the linear case discussed in Section 3, we get a number of results as corollaries of Theorem 7, and Lemmas 7 and 6.

Corollary 1. *Let $\mathcal{M}^{SA}$ be a supply-aware linear exchange market.*

- *The set of equilibrium prices of a market $\mathcal{M}^{SA}$ form a connected set.*
- *If the corresponding linear exchange market $\mathcal{M}$ has a unique CE prices where every good is liked by at least two agents, then market $\mathcal{M}^{SA}$ also has a unique equilibrium.*

The computation of a competitive equilibrium of an exchange market with separable piecewise linear concave (PLC) function is PPAD-hard, even when the PLC function for each agent-good pair has exactly two segments, with zero slope for the second segment [9]. Further, the set of competitive equilibrium prices of these markets can be disconnected. Note that, supply-aware exchange market with linear utilities is a special case of this market where the second segment starts at amount equal to the total supply of the respective good in each function. Corollary 1 shows that this restriction surprisingly makes the market well behaved, in the sense that the set of CE prices is connected, and equilibrium computation is efficient because it suffices to find a CE of linear exchange market [20].

6 Discussion

Walras designed the tatonnement process of price adjustment, where prices of goods with excess demand are increased and those with excess supply are decreased, until the market reaches an equilibrium. There has been much work analyzing convergence of the variants of tatonnement process in exchange markets, with positive results for markets satisfying weak-gross substitute property (which includes linear markets) [4,30,10]. We saw that (Example 4) even supply-aware *linear* market is not WGS, and preliminary investigation shows that such a process is locally divergent in them. It would be interesting to know if any variant of tatonnement converges, and to understand the special properties of the convergent point. There are a whole lot of questions on the computational aspect of supply-aware market, like sufficiency conditions for the existence of equilibria (see Example 3), efficient algorithms or hardness in case of SPLC/Leontief/CES/PLC utilities (see Example 2).

In the exchange market game, non-symmetric Nash equilibria remains to be understood. The current analysis of general exchange market game restricts the strategy set of agents to linear functions. It is unclear how the Nash equilibria behave if concave functions are also allowed. As in [7,8] for Fisher markets, it would be interesting to obtain an upper bound on the amount of gain an agent can ensure by strategizing her utility functions in exchange markets.

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A Proofs

Proof of Lemma 4:

Let $\boldsymbol{p}^*$ be a CE prices of market $\mathcal{M}$, and $\boldsymbol{x}^*$ be an optimal allocation at $\boldsymbol{p}^*$. Set $\boldsymbol{s}_i = \boldsymbol{p}^*, \forall i \in \mathcal{A}$. The only CE prices for $\boldsymbol{s}$ is again $\boldsymbol{p}^*$. Since, allocation $\boldsymbol{x}^*$ satisfies market clearing conditions (2) at $\boldsymbol{p}^*$, it is a feasible allocation when $\boldsymbol{s}$ is played, i.e., $\boldsymbol{x}^* \in \mathcal{X}(\boldsymbol{s})$. Further, $\boldsymbol{x}^*$ being a CE allocation, every agent receives an optimal bundle in it, hence it is also conflict-free. Thus $\boldsymbol{s}$ is a symmetric, conflict-free strategy profile and hence is a Nash equilibrium of $\Gamma(\mathcal{M})$ (Theorem 4). □

Proof of Lemma 5:

Since, $\boldsymbol{s}$ is symmetric, corresponding CE prices are $\boldsymbol{p}' = \boldsymbol{s}'_i$ (Lemma 1), and hence $\mathcal{X}(\boldsymbol{s}') = \{\boldsymbol{x} \mid \sum_i x_{ij} = 1,\ \forall j \in \mathcal{G};\ \ \boldsymbol{x}_i \cdot \boldsymbol{p}' \le \boldsymbol{w}_i \cdot \boldsymbol{p}',\ \forall i \in \mathcal{A}\}$. Further, $\boldsymbol{x}'$ is a conflict-free allocation of set $\mathcal{X}(\boldsymbol{s})$. Suppose, $U' = (U_1(\boldsymbol{x}'_1), \ldots, U_m(\boldsymbol{x}'_m))$ is not Pareto optimal. Then there exists $U^* \in \mathcal{U}$ with $U^*_i \ge U_i, \forall i$ and strict inequality for some agent, say k. Let the allocation achieving U^* be $\boldsymbol{x}^*$.

Since agent k likes $\boldsymbol{x}'_k$ the best among all of $\mathcal{X}(\boldsymbol{s}')$, it should be the case that she can't afford bundle $\boldsymbol{x}^*_k$ at prices $\boldsymbol{p}'$. So, we have $\boldsymbol{x}^*_i \cdot \boldsymbol{p}' > \boldsymbol{x}'_i \cdot \boldsymbol{p}'$, and since total money in the market is $\sum_{ij} w_{ij} p'_j = \sum_j p'_j = 1$, we also have $\sum_{i \ne k} \boldsymbol{x}^*_i \cdot \boldsymbol{p}' <$

$\sum_{i \neq k} \boldsymbol{x}'_i \cdot \boldsymbol{p}'$. This implies that at $\boldsymbol{x}^*$ the agents, except k, can achieve at least as much as at $\boldsymbol{x}'$ by cumulatively spending less. In that case, $\exists i \neq k$ such that she can get a better bundle than $\boldsymbol{x}'_i$ at prices $\boldsymbol{p}'$, i.e., $\pi_i(\boldsymbol{s}') > U_i(\boldsymbol{x}'_i)$, contradicting conflict-freeness of $\boldsymbol{s}'$. □

Proof of Lemma 6:

First, let us assume that $w_{ij} > 0, \forall i \in \mathcal{A}, \forall j \in \mathcal{G}$, to get rid of the trivial cases. Let $\boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_m)$ be the profile corresponding to true utilities.

If there is exactly one symmetric NE then market $\mathcal{M}$ has to have a unique CE prices (Lemma 4). Let these be $\boldsymbol{p}^*$. If degree of a good, say l, is one in $G(\boldsymbol{u}, \boldsymbol{p}^*)$, then we will construct more symmetric NE. Let the only edge of good l be from agent k. It is easy to check that there exist $\alpha < 1$ and $\beta > 1$, such that strategy profile $\boldsymbol{s}'$, where $\boldsymbol{s}'_i = (\beta p^*_1, \dots, \beta p^*_{l-1}, \alpha p^*_l, \beta p{*}_{l+1}, \dots, \beta p{*}_n)$ and $\sum_{j \in \mathcal{G}} s'_{ij} = 1$, for all $i \in \mathcal{A}$, is a symmetric NE.

For the other direction, we show that the only symmetric NE is $\boldsymbol{s}$, where $\boldsymbol{s}_i = \boldsymbol{p}^*$, $\forall i \in \mathcal{A}$. Suppose, there exists another symmetric NE $\boldsymbol{s}' \neq \boldsymbol{s}$. Let $\boldsymbol{p}' = \boldsymbol{s}'_i$ be the CE prices of the corresponding played market (Lemma 1). Consider the set $J = \{j \in \mathcal{G} \mid {}^{p'_j}/_{p^*_j} = \min_l {}^{p'_l}/_{p^*_l}\}$. Let $N(J)$ be the set of agents connected to J in graph $G(\boldsymbol{u}, \boldsymbol{p}^*)$. Note that, these connections will remain intact in $G(\boldsymbol{u}, \boldsymbol{p}')$ since while going form $\boldsymbol{p}^*$ to $\boldsymbol{p}'$ prices of goods in J gets scaled by the same factor.

At prices $\boldsymbol{p}'$ every agent in $N(J)$ wishes to get only goods of J. Total earnings of agents of $N(J)$ is more than the total prices of goods in J (since all w_{ij}'s are positive). Therefore, it is easy to check that if the degree of every good in J is more than one, then it will create conflict among the agents of $N(J)$ for the allocation, and hence $\boldsymbol{s}'$ is not a symmetric NE (Theorem 4). □

Proof of Lemma 9:

Let $\boldsymbol{p}^*$ and $\boldsymbol{x}^*$ be a competitive equilibrium prices and allocation of market $\mathcal{M}$. Consider a symmetric strategy profile $\boldsymbol{s}$ where $\boldsymbol{s}_i = \boldsymbol{p}^*, \forall i$. The CE prices in the played market for $\boldsymbol{s}$ will be $\boldsymbol{p}^*$. Further, $\boldsymbol{x}^* \in \mathcal{X}(\boldsymbol{s})$ and it is conflict-free. Thus $\boldsymbol{s}$ is symmetric NE (due to Lemma 8). □

Proof of Lemma 10:

Let $\boldsymbol{s}'$ be a symmetric Nash equilibrium with conflict-free allocation $\boldsymbol{x}' = \boldsymbol{x}(\boldsymbol{s}')$. We need to show that point $U' = (U_1(\boldsymbol{x}'_1), \dots, U_m(\boldsymbol{x}'_m))$ is Pareto-optimal in $\mathcal{U}$. Suppose not, and let $U^* \in \mathcal{U}$ is a better point and let $\boldsymbol{x}^*$ be the corresponding allocation. The proof is based on the same intuition as the proof of Lemma 5.

Suppose, agent k gets more in U^* than in U'. Then clearly, at CE prices $\boldsymbol{p}' = \boldsymbol{s}'_i$ of the played market for $\boldsymbol{s}'$ (Lemma 1), she can't afford bundle $\boldsymbol{x}^*_k$, *i.e.*, $\boldsymbol{x}^*_k \cdot \boldsymbol{p}' > \boldsymbol{x}'_k \cdot \boldsymbol{p}'$. Since, total money in the market is $\sum_{i,j} w_i j p'_j = \sum_j p'_j = 1$ (a constant), rest of the agents gets at least as much utility from $\boldsymbol{x}^*$ as from $\boldsymbol{x}'$, while spending strictly less. In that case, $\boldsymbol{x}'$ can not be conflict-free for them. □

B Proof of the Connectedness Lemma

Let $\mathcal{M}$ be a linear exchange market defined in Section 2. Let S be the symmetric Nash equilibria (SNE) of game $\Gamma(\mathcal{M})$, and let S^{CE} be those achieving competitive equilibrium (CE) prices and allocations (Lemma 4). Since, set of CE prices of market $\mathcal{M}$ is convex [16], so is set S^{CE}. We show that every point of S is connected to some point of S^{CE}.

Given a symmetric profile $\boldsymbol{s}$ and the CE prices $\boldsymbol{p}$ of its played market, we have $\boldsymbol{s}_i = \boldsymbol{p}, \forall i$ (Lemma 1). Therefore, $\boldsymbol{p}$ is enough to represent the profile $\boldsymbol{s}$. Henceforth, S and S^{NE} be the sets of price vectors achieved at corresponding SNEs. For a SNE prices $\boldsymbol{p} \in S$, define $\boldsymbol{x}(\boldsymbol{p})$ as,

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{x}(\boldsymbol{s}), \quad \text{where} \quad \boldsymbol{s}_i = \boldsymbol{p},\ \forall i$$

Claim. Let $\boldsymbol{s}$ be a symmetric profile, and let $\boldsymbol{p}$ be the corresponding CE prices. For an allocation $\boldsymbol{x}_i$ if $\boldsymbol{x}_i \cdot \boldsymbol{p} = \boldsymbol{w}_i \cdot \boldsymbol{p}$, then $\exists \boldsymbol{x}' \in \mathcal{X}(\boldsymbol{s})$ such that $\boldsymbol{x}_i = \boldsymbol{x}'_i$.

Proof. The proof follows using the fact that $G(\boldsymbol{s}, \boldsymbol{p})$, where an edge between agent i and good j indicates any amount of good j can be allocated to agent i, is a complete bipartite graph.

For SNE prices $\boldsymbol{p} \in S$, agent i derives utility ${u_{ij}}/{p_j}$ by spending a unit money on good j at these prices. Let this ratio be called bang-per-buck of agent i for good j. Since, earnings of agent i is limited, ideally she would want to buy goods in decreasing order of bang-per-buck until her money runs out. Using the above claim, it is easy to see that the conflict-free allocation $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p})$ achieves this ideal assignment for all the agents. Formally,

$$\begin{array}{ll} x_{ij} > 0, \frac{u_{ij}}{p_j} < \frac{u_{ij'}}{p_{j'}} & \Rightarrow x_{ij'} = 1 \\ x_{ij} = 0, x_{ij'} > 0 \Rightarrow \frac{u_{ij}}{p_j} \le \frac{u_{ij'}}{p_{j'}} & \end{array} \tag{7}$$

The only difference between a CE allocation and SNE allocation is that, in CE an agent buys only her maximum bang-per-buck goods (see (1)), while in SNE she buys lower bang-per-buck goods as well, but only after consuming the higher bang-per-buck goods completely. Let these higher bang-per-buck goods that she buys completely be called her *forced goods*, i.e., j' in the first condition of (7).

Now, consider a SNE prices $\boldsymbol{p}$, and its outcome allocation $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p})$. Let $\mathcal{G}^e(\boldsymbol{p})$ be the set of forced goods, i.e., Using (1), (7) and the fact that both CE and SNE prices and allocation have to satisfy market clearing conditions (2), it follows that if $\mathcal{G}^e(\boldsymbol{p})$ is empty, then $\boldsymbol{p}$ is a CE prices and $\boldsymbol{s} \in S^{NE}$.

For the other case, we will reduce the size of set $\mathcal{G}^e(\boldsymbol{p})$ inductively. For now let us consider the Fisher setting by assuming $w_{ij} = w_i,\ \forall(i,j)$, where $w_i > 0, \forall i$ and $\sum_i w_i = 1$, and later show how to handle the general case. In such a market, earnings of agent i remains w_i at any prices $\boldsymbol{p}$ summing to one.

Consider a SNE prices $\boldsymbol{p} \in S$, and let $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p})$ be the corresponding outcome allocation. From the above discussion, we know that they satisfy (2). Next we will update $(\boldsymbol{p}, \boldsymbol{x})$ so that (2) is always satisfied, thereby ensuring that $\boldsymbol{p}$ remains in S, while reducing the size of $\mathcal{G}^e(\boldsymbol{p})$.

Define a bipartite graph $G(\boldsymbol{x}, \boldsymbol{p})$ between agent and goods nodes as follows: There is an edge (i, j) in the graph if either $x_{ij} > 0$, or $x_{ij'} > 0$ and $u_{ij}/p_j = u_{ij'}/p_{j'}$. An *alternating path* in this graph, from node a to node b is the one where odd level edges from node a have non-zero x_{ij}. Wlog we assume that for any $\boldsymbol{p} \in S$, corresponding $\boldsymbol{x}(\boldsymbol{p})$ forms a forest.

1. If $\mathcal{G}^e(\boldsymbol{p}) = \emptyset$ then STOP.
2. Let $l \in \mathcal{G}^e(\boldsymbol{p})$ be the forced good of agent k with least bang-per-buck among all her forced goods.
3. Let T be the subgraph of $G(\boldsymbol{x}, \boldsymbol{p})$ containing node k and all the alternating paths from k, and let N be the set of goods in T except for those in $\mathcal{G}^e(\boldsymbol{p})$.
4. Let $p_l = \alpha p_l$, $\forall j \in N, p_j = \beta p_j$, where $\beta = \alpha p_k / \sum_{j \in N} p_j$. the set of nodes reachable from node l in graph $G(\boldsymbol{x}, \boldsymbol{p})$ through an alternating path. Clearly, if $\alpha > 1$ then $\beta < 1$.
5. Starting from $\alpha = 1$, increase it, and accordingly change $\boldsymbol{x}$ so that it remains $\boldsymbol{x}(\boldsymbol{p})$. This can be done by increasing the money flow on edge (k, l) and on even level edges of T from node k, and decreasing on its odd level edges from k. Do this until one of the following happens.
   - If for good $j \in N$ adjacent to k in T, we get $u_{kl}/p_l = u_{kj}/p_j$, then remove l from $\mathcal{G}^e(\boldsymbol{p})$ and go to Step 1, i.e., l is no more a forced good.
   - If x_{ij} becomes zero, then do the following: If $i \neq k$ then recalculate $G(\boldsymbol{x}, \boldsymbol{p})$ and go to Step 3 (edge (i, j) will be removed in T). Otherwise, if $x_{kj} = 0, \forall j \in N$, then remove l from $\mathcal{G}^e(\boldsymbol{p})$ and go to Step 1.
   - If an agent $i \notin T$ gets interested in a good $j \in N$, *i.e.,* her bang-per-buck for good j becomes same as the bang-per-buck of a good she is buying, then recalculate $G(\boldsymbol{x}, \boldsymbol{p})$ and go to Step 3.

The above procedure is similar to the one used in [18] for equilibrium computation in Fisher markets. Due to the claim, when the above procedure terminates we have $\boldsymbol{p} \in S^{NE}$. Since, $\boldsymbol{p} \in S$ through out the procedure, we get that S is connected.

The above procedure tries to increase the price of good l, to make it a non-forced good of agent k, while maintaining that $\boldsymbol{p}$ remains a SNE price and $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p})$. Therefore, it has to decrease prices in T by a same factor. When this is done, prices in other components of $G(\boldsymbol{p}, \boldsymbol{x})$ do not change because the earnings of all the agents are fixed (w_i for agent i).

If we consider arbitrary exchange market, where w_{ij}s of an agent are not same, then the earnings of agent i is $\sum_{ij} w_{ij} p_j$, and hence will change with prices. In that case, prices of goods in components other than T will also change, which will have to be taken care of. We will hold prices of goods in $\mathcal{G}^e(\boldsymbol{p})$ as they are, except for good l. For the rest, in a component other than T, prices of all the goods, except for those in $\mathcal{G}^e(\boldsymbol{p})$, will change by the same factor, say γ. It can be shown that γ has to be a convex combination of α and β.

In the last case of Step 5, it is assumed that $i \notin T$ can not get interested in a good of $\mathcal{G}^e(\boldsymbol{p}) \cap T$. This is true under the previous assumption, because prices are fixed outside T. If prices in a component other than T are increasing ($\gamma > 1$), then its agent can get interested in a good of $j \in \mathcal{G}^e(\boldsymbol{p})$. In that case we switch reset $l = j$ and restart from Step 2, where then l is not changed. After this price is good j is going to increase, and agent will be no more interested in it.

The rest remains the same. The procedure stops in finite number of steps, because one can show that the difference between ratios u_{kl}/p_l and u_{kj}/p_j for $(k, j) \in T$ decreases by at least an absolute constant. And good l becomes non-forced as soon as (or before) they become equal.