1 Formulation

An equilibrium problem on a transport network

Consider the following problem of determining fair prices for origin destination flows on a transport network. The network consists of n nodes and we may assume that there are links connecting all pairs of nodes. These links represent the (best) route from origin to destination, and although a path between two nodes may pass through other nodes in practice, for this problem, the complete graph on n vertices is more convenient to start with.

Demand for transport arises in the following way. There are a large number of independent customers who express their demand (in truck loads, for convenience). This can be approximated as a Poisson demand with a rate Λ_{ij} for demand from origin i to destination j. Note that the demand matrix Λ will not be symmetric in general. In fact, part of this exercise will be to explicitly explain the prevalence of non-symmetric transport rates in practice.

Costs are modelled as follows. For every pair $i-j$, there is a time of transit t_{ij} and a proportional cost per truck per unit time d, which captures the variable costs of shipment. There is a time cost of operating a truck, taken as h. Initially, we ignore the fixed cost component of movement, and assume that all shipments are in truck loads. In addition, there is a premium c_{ij} which is charged on the $i-j$ sector. This is what has to be determined at equilibrium. Equilibrium is defined in the following way.

Define a strategy of a (monopolistic) transporter as follows. Given premiums c_{ij} and the demand rate matrix Λ and the times of movement, a strategy is a collection of sets A_i (defined at node i) which define the destinations which the transporter will accept at node i. Given a strategy, the transporter will operate between locations as specified by the sets A_i. The movements of a single truck can then be assumed to follow the states covered by the strategy sets A_i and one can probabilistically estimate the expected return on each truck. In general, the truck waits (at location i) for traffic (determined by the strategy set A_i and the demand rates), and then does a loaded run which earns revenue.

Suppose a truck begins at location i_1 and visits n locations $i_1, ..., i_n$ in that order. The expected time spent in the transition from i_r to i_{r+1} is

$$T(i_r, i_{r+1}) = w_r + t_{i_r+i_{r=1}},$$

where w_r is the expected waiting time at location r. The waiting time depends on the strategy at r and the rates of demand from r tolocations in

A_r. The costs incurred have the components h times the total time in the system and d times the running time. A trip from i to j earns c_{ij}.

In the long run, the total revenue ratio is

$$\lim_{n\to\infty} E \sum_{k=1}^{n} \frac{c_{i_k, i_{k+1}}}{hT(i_r, i_{r+1}) + dt_{i_r + i_{r=1}}}.$$

This can be evaluated for various strategy choices A_i. One hypothesis worth examining is that for a given set of prices c_{ij}, the strategy A_i is common for all i; that is, there is a common set of desirable locations that all traffic aims at catering to.

For a reasonable objective in terms of this revenue ratio, the equilibrium problem is to determine values of c_{ij} which will see that A_i at every location is the set of all locations (i.e. all demand is met, with appropriate "fair" prices).

The macro picture

Given the transport demand rate matrix Λ, we can infer that there are certain locations that are overall sources of transport demand (say N of them, and some locations (say M of them) that are sinks. A network flow model (the transportation model, on N origin nodes and M destination nodes with an appropriate cost minimising objective function based on distances between nodes) would give the flows of empty vehicles so that a cyclic flow of vehicles is maintained in the long run.

Therefore, from a deficit location, some number of empty movements (to various surplus locations) would be there in the long run. From deficit location i, let p_i be the fraction of all vehicles that would ply empty from i and let p_{ij} be the fraction of vehicles that ply empty to j from i. The modified return function, therefore has to be derived, also in terms of p_{ij} values. This would now account for the fact that transport rates from i to j would eventually reflect "true" costs from i to j plus the attractiveness of j as a location for picking up further traffic.

2 A Simple Formulation

A simple variation of the trucker's problem is the 'ticket-checker's problem'. There is a collection of terminals $N = \{1, 2, \ldots, n\}$. The bus-system arranges buses between every pair of destinations. For every pair i, j, there is a rate λ_{ij}, such that buses leave for j from i, in a Poisson process with rate λ_{ij}. The ticket-checker's (TC) problem is to travel on these buses, checking tickets, from terminal to terminal. The policy of the TC is to reach a particular destination, and take the first bus out of it, and so on. The earning rate of the TC is the fraction of the time she spends on the bus, with respect to the total elapsed time.

The two data items are the matrix $\Lambda = (\lambda_{ij})$, and $T = (t_{ij})$, where λ_{ij} is as above, and t_{ij} is a fixed number denoting the travel time from i to j. Based on this data, we may define a transition system $P = (p_{ij})$, where $p_{ij} = \frac{\lambda_{ij}}{\sum_k \lambda_{ik}}$.

Let us denote $\sum_k \lambda_{ik}$ by μ_i. Clearly, the expected waiting time for the TC at terminal i is $w_i = \frac{1}{\mu_i}$. Let π be the steady state for the chain P, *i.e.*, $\pi P = \pi$. The earning rate $ER(\Lambda, T)$ is given by:

$$ER(\Lambda, T) = \frac{\sum_{i,j} \pi_i p_{ij} t_{ij}}{\sum_{i,j} \pi_i p_{ij} (w_i + t_{ij})}$$

3 Strategies

The earning rate above is for the simple strategy of hopping on to the next bus (for whichever destination) starting at the TC's current destination.

We next propose two possible options which the TC may exercise:

1. **Refusal**: The TC may refuse to board a particular bus, and wait for the next starting bus.

2. **Empty Travel**: The TC may travel by other means from i to j, there-by checking no tickets on this part of the journey.

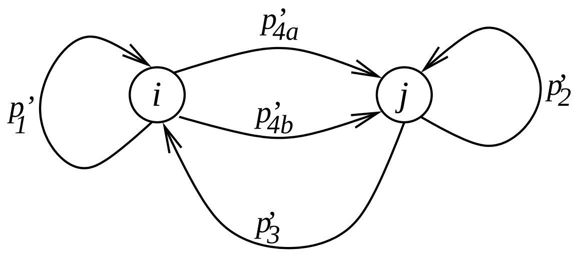
3.1 Refusal

We model **refusal** by assuming that the TC follows a rate-matrix $\Lambda' = (\lambda'_{ij})$ such that $0 \leq \lambda'_{ij} \leq \lambda_{ij}$. Thus, the TC may, optionally reduce the arrival rate of the bus-service from i to j, and in the limit, may completely ignore buses from i to j. The first question is whether $ER(\Lambda', T) > ER(\Lambda, T)$ for a suitable Λ' as above.

To begin with, let us assume that there are fixed i and j, and we shall examine $\Lambda' = \Lambda - \alpha E_{ij}$, where E_{ij} is the matrix with 1 in the i-j-th place and 0 everywhere else. Next, for this fixed i, j, we set up an associated markov chain on two states, *viz.*, $\{i, j\}$. For every run $\sigma = (\sigma_0, \sigma_1, \ldots,)$ of the original transition system, we produce a run $\beta = (\beta_0, \beta_1, \ldots)$, on the reduced system Q' as follows: We assume that $\sigma_0 = i$, and that $0 = i_0 < i_1 \ldots$ are all the indices r such that $\sigma_r \in \{i, j\}$. The run β is defined as $\beta_k = \sigma_{i_k}$. Note that there are five types of transitions in the system Q' which are enumerated below:

1. $\beta_k = i$ and $\beta_{k+1} = i$.
2. $\beta_k = j$ and $\beta_{k+1} = j$.
3. $\beta_k = j$ and $\beta_{k+1} = i$.
4. $\beta_k = i$ and $\beta_{k+1} = j$. This may be further split into
   (a) $i_{k+1} = i_k + 1$, and thus σ has directly moved from i to j.
   (b) $i_{k+1} > i_k + 1$, and thus σ has moved to some other destination from i before visiting j.

We collect the above transitions in an indexing set $S' = \{1, 2, 3, 4a, 4b\}$. For the data (Λ, T), we may define the quantities $(p_s)_{s \in S'}, (t_s)_{s \in S'}$ and $(e_s)_{s \in S'}$, where p_s is the probability of making the transition s from the current destination, t_s is the expecting busing time for the transition and e_s is the expected duration of the transition, which includes waiting times at intermediate destinations.



It may be argued that the transition system Q' represents P as far as aggregate earning rates are concerned.

Let $\overline{\pi}$ be the steady-state for the system Q'. The earning rate may be writen in terms of Q', as follows:

$$ER(\Lambda, T) = \frac{\overline{\pi}(i)p_1 t_1 + \overline{\pi}(i)p_{4a}t_{4a} + \overline{\pi}(i)p_{4b}t_{4b} + \overline{\pi}(j)p_2 t_2 + \overline{\pi}(j)p_3 t_3}{\overline{\pi}(i)p_1 e_1 + \overline{\pi}(i)p_{4a}e_{4a} + \overline{\pi}(i)p_{4b}e_{4b} + \overline{\pi}(j)p_2 e_2 + \overline{\pi}(j)p_3 e_3}$$

The same calculation may be done for $\Lambda(\alpha) = \Lambda' = \Lambda - \alpha E_{ij}$. The associated variables are $\{p'_s, e'_s, t'_s | s \in S'\}$. However, note that

- $p'_2 = p_2$, $p'_3 = p_3$.

- $e'_2 = e_2$, $e'_3 = e_3$.
- $t'_s = t_s$ for all $s \in S'$.

We may thus form the function:

$$ER_r(\Lambda(\alpha), T) = \frac{\overline{\pi}'(i)p'_1 t_1 + \overline{\pi}'(i)p'_{4a}t_{4a} + \overline{\pi}'(i)p'_{4b}t_{4b} + \overline{\pi}'(j)p'_2 t_2 + \overline{\pi}'(j)p'_3 t_3}{\overline{\pi}'(i)p'_1 e'_1 + \overline{\pi}'(i)p'_{4a}e'_{4a} + \overline{\pi}'(i)p'_{4b}e'_{4b} + \overline{\pi}'(j)p'_2 e'_2 + \overline{\pi}'(j)p'_3 e'_3}$$

Clearly, $ER(\Lambda, T) = ER_r(\Lambda(0), T)$. The first task is to understand the function above and properties with respect to α, such as (i) convexity, (ii) whether for **small** α, $ER_r(\Lambda(\alpha), T) > ER_r(\Lambda(0), T)$, and finally (iii) whether for some **large** α, with $0 \leq \alpha \leq \lambda_{ij}$, $ER_r(\Lambda(\alpha), T) > ER_r(\Lambda(0), T)$.

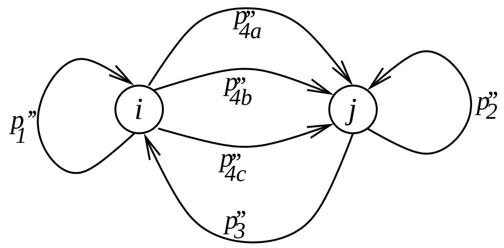
3.2 Empty Returns

We model empty returns as a new rate matrix $\Lambda'' = \Lambda + \delta E_{ij}$. If at i, an arrival for j corresponding to the additional rate δ takes place, then the TC rides this bus to j but without accruing the travel time t_{ij}. We thus have the corresponding markov chain Q'' again on the states $\{i, j\}$ and defined below.

For every run $\sigma = (\sigma_0, \sigma_1, \ldots,)$ of the original transition system, we produce a run $\beta = (\beta_0, \beta_1, \ldots)$, on the reduced system Q'' as follows: We assume that $\sigma_0 = i$, and that $0 = i_0 < i_1 \ldots$ are all the indices r such that $\sigma_r \in \{i, j\}$. The run β is defined as $\beta_k = \sigma_{i_k}$. Note that there are six types of transitions in the system Q' which are enumerated below:

1. $\beta_k = i$ and $\beta_{k+1} = i$.
2. $\beta_k = j$ and $\beta_{k+1} = j$.
3. $\beta_k = j$ and $\beta_{k+1} = i$.
4. $\beta_k = i$ and $\beta_{k+1} = j$. This may be further split into
   (a) $i_{k+1} = i_k + 1$, and σ has directly moved from i to j on an arrival from λ_{ij}.
   (b) $i_{k+1} = i_k + 1$, and thus σ has directly moved from i to j on an arrival from δ.
   (c) $i_{k+1} > i_k + 1$, and thus σ has moved to some other destination from i before visiting j.

We collect the above transitions in an indexing set $S'' = \{1, 2, 3, 4a, 4b, 4c\}$. For the data (Λ, T), we may define the quantities $(p''_s)_{s\in S''}, (t''_s)_{s\in S''}$ and $(e''_s)_{s\in S''}$, where p''_s is the probability of making the transition s from the current destination, t''_s is the expecting busing time for the transition and e''_s is the expected duration of the transition, which includes waiting times at intermediate destinations.



We may thus form the function:
$ER_e(\Lambda(\delta), T)$

$$= \frac{\pi''(i)p''_1 t_1 + \pi''(i)p''_{4a}t_{4a} + \pi''(i)p''_{4c}t_{4c} + \pi''(j)p''_2 t_2 + \pi''(j)p''_3 t_3}{\pi''(i)p''_1 e''_1 + \pi''(i)p''_{4a}e''_{4a} + \pi''(i)p''_{4b}e''_{4b} + \pi''(i)p''_{4c}e''_{4c} + \pi''(j)p''_2 e''_2 + \pi''(j)p''_3 e''_3}$$

Note that the numerator does not contain the term for the transition $4b$ which corresponds to empty movement, while it is charged in the elapsed time in the denominator. Clearly, $ER_e(\Lambda(0), T) = ER(\Lambda, T)$. We ask the same questions as in the refusal function ER_r.

4 The Competeing Truck Problem

We have seen earlier that the one-truck earning rate problem has a simple and exact formulation when arrivals are Poisson. We have also seen various strategies that this single truck may employ to maximize its returns. In this section we outline the situation where many trucks, say M, compete in the same customer traffic scenario.

Thus, in other words, there is the same rate matrix Λ and the transit time matrix T. However, there are now M trucks which are competeing for providing service to these customer arrivals.

Our first task will be to outline a finite-state transition system modelling this process. We shall see that the exact model will be (within the authors' limitations) computationally intractable. To illustrate this, we will choose a two destination set $N = \{1, 2\}$. We may assume for simplicity, either that, at a destination, when a customer arrives, she chooses randomly a truck which is waiting at that destination, or that the trucks queue up and she chooses the one at the head of the queue. Clearly, the aggregate earning measures are the same in either policy.

For the two destination case, a state of the system is parametrized be the 4-tuple $(n_1, n_{12}, n_2, n_{21})$, where n_i indicates the trucks waiting at destination i, and n_{ij} indicates the trucks *en route* from i to j. The state space $\mathcal{S}(M)$ may be defined as:

$$\mathcal{S}(M) = \{(n_1, n_{12}, n_2, n_{21} | n_1 + n_2 + n_{12} + n_{21} = M\}$$

Given a state $\overline{n} = (n_1, n_{12}, n_2, n_{21})$, we define a transition to $\overline{n}' = (n_1', n_{12}', n_2', n_{21}')$, to mark either the arrival of a truck at a destination, or the arrival of a customer and thus the departure of a truck from a destination. For example, when a customer arrives at destination 1, the next state $\overline{n}'$ is given by $\overline{n}' = (n_1 - 1, n_{12} + 1, n_2, n_{21})$. This transition system is a simple markov chain if we assume, for example, that the inter-destination transit times are also memory-less with means (t_{ij}). Even with this assumption, this chain seems compuationally intractable and unable to give us a qualititative understanding of the effect of competition on the earning rates of individual trucks.

We now proceed in another way. We install an observer on the road from i to j and measure the rate θ_{ij} of the number of trucks per unit time which travel on this road. Equipped with these numbers, we assign $\theta_i = \sum_k \theta_{ik}$ as the net departure, and therefore the net arrival rate of trucks at i. Since customers arrive independently as a Poisson process, we see that $\theta_{ij} = p_{ij}\theta_i$, where $p_{ij} = \frac{\lambda_{ij}}{\mu_i}$ and $\mu_i = \sum_k \lambda_{ik}$ is the probability that a customer arrival at i is for j. By the conservation of trucks, we have the **Rate Equation**:

$$\sum_i \theta_i p_{ij} = \theta_j$$

In other words, if $P = (p_{ij})$ is the matrix as defined earlier, then $\theta P = \theta$ and thus the vector θ is a multiple of the steady state π of P. In other words, there is a $\gamma > 0$ such that $\theta = \gamma\pi$.

There is now the temptation to assume, what we call the **Poisson assumption**, that the trucks actually arrive at each i as a Poisson process with rate θ_i. This belief (although inaccurate) is strengthened by the local facts, that (i) if indeed trucks were to arrive at i in a Poisson process with rate θ_i, then they would depart in *mutually independent* Poisson processes with rates (θ_{ij}), and (ii) if trucks were to arrive at i from k in mutually independent Poisson processes with rates (θ_{ki}), then the net arrival process at i would indeed be Poisson with rate θ_i.

Under the Poisson assumption, the analysis becomes very simple: Each destination serves as an $M/M/1$ queue with memory-less arrivals (of trucks) with rate θ_i and memory-less service-times (by customers) with rate μ_i. Now, the average length of the queue at i equals $\frac{\theta_i}{\mu_i - \theta_i}$. Since a truck is either in a queue, or *en route*, we have the following **truck inventory equation**:

$$\sum_i \frac{\theta_i}{\mu_i - \theta_i} + \sum_{i,j} \theta_i p_{ij} t_{ij} = M$$

Since $\theta = \gamma\pi$, the only unknown in the above equation is the multiplier γ, which may now be determined. By Little's theorem, the earning rate $E(\Gamma, T, M)$ is given by the fraction of trucks in motion, and thus:

$$E(\Gamma, T, M) = \frac{\sum_{i,j} \theta_i p_{ij} t_{ij}}{\sum_i \frac{\theta_i}{\mu_i - \theta_i} + \sum_{i,j} \theta_i p_{ij} t_{ij}} = \frac{\sum_{i,j} \theta_i p_{ij} t_{ij}}{M}$$

It is instructive to compare $E(\Lambda, T, 1)$ with $E(\Lambda, T)$ of the previous sections. For small θ_i's, we see that

$$E(\Gamma, T, M) \approx \frac{\sum_{i,j} \theta_i p_{ij} t_{ij}}{\sum_i \frac{\theta_i}{\mu_i} + \sum_{i,j} \theta_i p_{ij} t_{ij}}$$

Now, since both numerator and denominator are linear in the θ_i's, we have:

$$E(\Gamma, T, M) \approx \frac{\sum_{i,j} \pi_i p_{ij} t_{ij}}{\sum_i \frac{\pi_i}{\mu_i} + \sum_{i,j} \pi_i p_{ij} t_{ij}} = E(\Lambda, T)$$

Clearly, we always have $E(\Lambda, T, 1) \leq E(\Lambda, T)$, with the difference attributable to the fact that in the computation of $E(\Lambda, T)$, the single truck always sees an empty queue at every destination.

We will now illustrate these calculations for a two destination system $N = \{1, 2\}$ with

$$\Lambda = \begin{bmatrix} 0 & 20 \\ 30 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus $\lambda_{12} = 20/hr$ and $t_{12} = 2hrs$. For the chain P, clearly $\pi = [\frac{1}{2}, \frac{1}{2}]$ is the steady state. Thus $\theta_{12} = \theta_{21}$, which is as should be, since trucks must be conserved. For a customer service rate of $\theta_{12} = \theta_{21} = 8$, we see that

$$M_8 = \frac{8}{20-8} + \frac{8}{30-8} + 8 \cdot 2 + 8 \cdot 3 = 41.030$$

For a customer service rate of $16/hr$, the same claculation becomes:

$$M_{16} = \frac{16}{20-16} + \frac{16}{30-16} + 16 \cdot 2 + 16 \cdot 3 = 85.142$$

Thus doubling the customer satisfaction rate from $8/hr$ to $16/hr$, requires a doubling of the number of trucks plying *and some more.* M_{19} is 115.727, while $M_{19.5}$ is a huge 138.357. Another option for the trucks would be to return empty from destination 1. Thus if $24/hr$ trucks went from 2 to 1, of which 6 returned empty and 18 queued up for customers, we would have:

$$M = \frac{18}{20-18} + \frac{24}{30-24} + 24 \cdot 2 + 24 \cdot 3 = 133$$

The expected number of revenue-earning trucks at any moment is $18 \cdot 2 + 24 \cdot 3 = 108$.

The above numbers have been tabulated below. Here δ_{12} is the **empty-return** rate, and M and E are the number of trucks and the earning rate, respectively. :

θ_{21}	$\theta_{12} - \delta_{12}$	δ_{12}	M	E
8	8	0	41.030	0.9748
16	16	0	85.42	0.9398
19	19	0	115.727	0.8209
19.5	19.5	0	138.357	0.7047
24	18	6	133	0.8120

The calculations thus seem to indicate that when roughly 135 trucks are plying the route, empty returns is a very profitable strategy. Similarly, there are situations where customer refusal also is a profitable strategy. We also see that, unlike in the single truck case, there are 'mixed' strategies which must be optimal, as is illustrated in the two-destination case.

5 Reluctance

In this section, we propose a rather simple model of capital depreciation. In our earlier analysis, once an optimal strategy, mixed or pure, has been found, the description of destinations as 'favourable' or 'unfavourable' should ideally cease. Under the adopted strategy, the truck operator should accept with equanimity, the customer which comes her way, saying to herself that 'eventualy things even out'.

We shall see, through our capital depreciation model, that 'unfavourable' destinations re-appear, and may lend an explanation of a possible reluctance of truck operators to go to 'unfavourable' locations.

For the same data (Λ, T, P), assume that we observe the truck in the state s, at the end of a hop. Then the trucking (busy) time for the following hop is given by

$$t(s) = \sum_{i,j} s(i) p_{ij} t_{ij}$$

Similarly, the elapse time for the hop is given by:

$$e(s) = \sum_{i} s(i) w(i) + \sum_{i,j} s(i) p_{ij} t_{ij}$$

We see that both $t(s)$ and $e(s)$ are linear functions on s.

Let us assume that there is a *per-hop depreciation rate* $0 < \rho \leq 1$. For the same data (Λ, T, P), assume that the truck begins its journey with an initial state s^0. As time progresses, after the k-th hop, the truck's state is clearly $s^0 P^k = s^k$, say. Thus, the expected total expected earnings of the truck operator after r hops is, in **today's** or **real** time/rupee terms $\sum_{i=0}^{r-1} t(s^i)\rho^i$, while the net expected elapsed time is $\sum_{i=0}^{r-1} e(s^i)\rho^i$.

Thus the expected total **real** trucking time $TRT(\Lambda, T, P, \rho, s)$ is given by:

$$TRT(\Lambda, T, P, \rho, s) = TRT(\rho, s) = \sum_{i=0}^{\infty} t(s^i)\rho^i$$

Similarly, the expected total real elapse time is given by:

$$TRE(\Lambda, T, P, \rho, s) = TRE(\rho, s) = \sum_{i=0}^{\infty} e(s^i)\rho^i$$

The next bit of simplification arises from expressing $s^0 = s$ as a linear combination of the eigen-vectors of P. So, let $Spec(P) = \{\kappa_1, \ldots, \kappa_N\}$ be the spectrum of P, with $\kappa_1 = 1$. Further, we assume that there are eigen-vectors x_r for each element κ_r of the spectrum. Recall that $x_1 = \pi$ is the steady state of P. Let $s = \sum_r c_r x_r$ be the expression of s as a linear combination of the eigen-vectors. We immediately see that:

$$s^i = \sum_r c_r \kappa_r^i x_r$$

Thus

$$TRT(\rho, s) = \sum_{i=0}^{\infty} t(s^i)\rho^i = \sum_r \sum_{i=0}^{\infty} c_r \kappa_r^i \rho^i t(x_r) = \sum_r c_r \frac{t(x_r)}{1 - \rho\kappa_r}$$

Recall that for any state s such that $\sum_i s(i) = 1$, we have $c_1 = 1$. Whence:

$$TRT(\rho, s) = \frac{t(\pi)}{1-\rho} + c_2 \frac{t(x_2)}{1 - \rho\kappa_2} + \ldots + c_n \frac{t(x_n)}{1 - \rho\kappa_n}$$

$$TRE(\rho, s) = \frac{e(\pi)}{1-\rho} + c_2 \frac{e(x_2)}{1 - \rho\kappa_2} + \ldots + c_n \frac{e(x_n)}{1 - \rho\kappa_n}$$

5.1 An Example

We will now illustrate these computations for our two destination system $N = \{1, 2\}$ with

$$\Lambda = \begin{bmatrix} 0 & 20 \\ 30 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We see that $Spec(P) = \{1, -1\}$ with $x_1 = [\frac{1}{2}, \frac{1}{2}]$ and $x_2 = [\frac{1}{2}, -\frac{1}{2}]$. Further, $t(x_1) = 2.5, t(x_2) = -0.5$, and $e(x_1) = 2.583, e(x_2) = -0.492$. For the pure states, we have

$$s_1 = [1, 0] = x_1 + x_2 \ \ and \ \ s_2 = [0, 1] = x_1 - x_2$$

For $\rho = 0.9$, we have the following table(where ER is the earning rate):

	TRT	TRE	ER
s_1	24.737	25.571	0.9674
s_2	25.263	26.089	0.9683

It is clear that if the truck operator had a choice to begin from any state, she would doubly prefer s_2: it has a *better* earning rate over a *longer* time.

6 Deadline Deaprtures

The next regime of operations we examine is what we call, the **deadline departures** policy. A truck, when arriving at destination i has a pre-destined scheduled departure time t_i. Upon arrival at destination i, it waits in a queue, behind trucks with earlier scheduled departure times. When it is at the head of the queue, before the time instant t_i if a customer arrives for destination j, the truck proceeds for j with a scheduled departure time of $t_i + t_{ij}$. However, if by t_i no customer arrives, then the truck departs for a pre-determined destination j_0 with a schduled departure at j_0 of $t_i + t_{ij_0}$.

We analyse here the behaviour of a simpler local version of this problem. Assume that trucks arrive at i at instants $\{\dots, -2R, -R, 0, R, 2R, \dots\}$. Furthermore, the schduled departure of the truck arriving at time nR is $nR+T$. Customers arrive in a Poisson process with rate λ. We wish to compute the probability α that a truck departs empty.

Let X_0 be the random variable denoting the departure time of the 0-th truck (which arrived at time $0R$), and X_1 the random variable such that $R + X_1$ is the departure time of the 1-st truck. Let f be the probability density function of X_0: it has support $[0, T]$. Note that f has an impulse of magnitude α to mark the probability of empty travel. In the steady state, f would also be the density of X_1.

Expressing the dependancy of X_1 on X_0, we see that if the 1-st truck has departed before the instant $T - R$, then either (i) the 0-th truck departs before the 1-st truck arrives, or else (ii) the 1-st truck has to wait till the 0-th truck departs and a customer arrives. Thus, we have for $t \in [0, T - R]$:

$$f(t) = \lambda e^{-\lambda t} \left[\int_0^R f(q) dq \right] + \int_R^{R+t} f(q) \lambda e^{-\lambda(t-q+R)} dq \tag{1}$$

Similarly, for $t \in [T - R, T]$, we have the third possibility that the 0-th truck departs empty:

$$f(t) = \lambda e^{-\lambda t} \left[\int_0^R f(q) dq \right] + \int_R^{T} f(q) \lambda e^{-\lambda(t-q+R)} dq + \alpha \lambda e^{-\lambda(t-T+R)} \tag{2}$$

Differentiating 1 with respect to t we see that:

$$f'(t) = -\lambda \left(\lambda e^{-\lambda t} \left[\int_0^R f(q) dq \right] \right) + \tag{3}$$

$$\frac{1}{\Delta t} \left[\int_R^{R+t+\Delta t} f(q) \lambda e^{-\lambda(t+\Delta t-q+R)} dq - \int_R^{R+t} f(q) \lambda e^{-\lambda(t-q+R)} dq \right] \tag{4}$$

the second term may be expanded as:

$$\frac{\int_{R+t}^{R+t+\Delta t} f(q)\lambda e^{-\lambda(t+\Delta t-q+R)}dq}{\Delta t}$$

$$+\frac{1}{\Delta t}\left[\int_{R}^{R+t} f(q)\lambda(e^{-\lambda(t+\Delta t-q+R)} - e^{-\lambda(t-q+R)})dq\right]$$

Since

$$f(R+t)\lambda = \frac{\int_{R+t}^{R+t+\Delta t} f(q)\lambda e^{-\lambda(t+\Delta t-q+R)}dq}{\Delta t}$$

we have:

$$f'(t) = -\lambda f(t) + \lambda f(R+t) \ \ for \ \ t \in [0, T-R]$$

On the other hand, for $t \in [T-R, T]$, we have:

$$f'(t) = -\lambda f(t) \ \ for \ \ t \in [T-R, T]$$