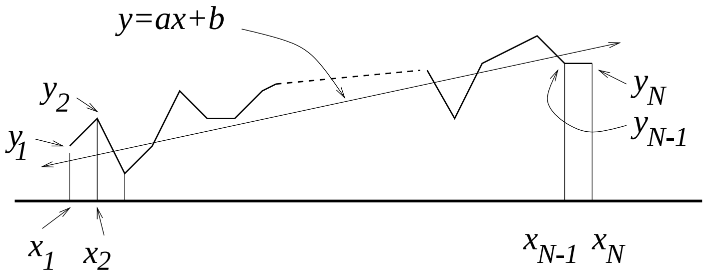

Figure 1: A Typical Scenario.

1 Fitting Geometry

1.1 Fitting Straight Lines

In this section, we consider the problem of fitting a straight line to given collection $P = \{(x_1, y_1), \ldots, (x_N, y_N)\}$ of N points in $\mathbb{R}^2$, the real plane. See Figure 1 for a typical scenario.

The first step is to consider the general line $y = ax + b$. For a sample datum (x_i, y_i), the ideal value of y is $ax_i + b$, in which case the line passes through this point. Whence a measure of the error for the i-th point, having chosen a, b is $(y_i - ax_i - b)^2$. We may write the total error as:

$$E(a, b) = \sum_{i=1}^{N} (y_i - ax_i - b)^2$$

We see that $E(a, b)$ is a polynomial in the variables a, b. We now get the conditions on a, b for $E(a, b)$ to be minimum. Clearly these are:

$$\begin{aligned} \frac{\partial E(a,b)}{\partial a} &= 0 \\ \frac{\partial E(a,b)}{\partial b} &= 0 \end{aligned}$$

We may simplify these to get:

$$\begin{aligned} \sum_{i=1}^{N} (y_i - ax_i - b) \cdot x_i &= 0 \\ \sum_{i=1}^{N} (y_i - ax_i - b) &= 0 \end{aligned}$$

Collecting terms we see that this gives us two linear equations in two unknowns:

$$\begin{aligned} a(\textstyle\sum_{i=1}^{N} x_i^2) \quad +b(\textstyle\sum_{i=1}^{N} x_i) &= \textstyle\sum_{i=1}^{N} x_i y_i \\ a(\textstyle\sum_{i=1}^{N}) \quad +bN &= \textstyle\sum_{i=1}^{N} y_i \end{aligned}$$

These may be solved to get the required values of a and b.

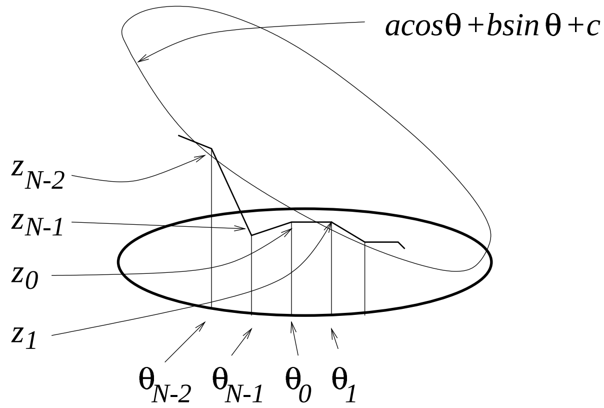

Figure 2: Data on a Circle.

1.2 Fitting to a Circle

In many situations, we have data on a circle. Thus $P = \{(\theta_0, z_0), \ldots, (\theta_{N-1}, z_{N-1}\}$, as shown in figure 2 below. The equivalent concept of a straight line is that of an ellipse given by

$$z = a\cos\theta + b\sin\theta + c$$

Again, we see up the error function:

$$E(a,b,c) = \sum_{i=0}^{N-1} (z_i - a\cos\theta_i - b\sin\theta_i - c)^2$$

As before, we may get the minimym error consition by differentiating $E(a,b,c)$ in the three variables as below:

$$\begin{aligned} \frac{\partial E(a,b,c)}{\partial a} &= 0 \\ \frac{\partial E(a,b,c)}{\partial b} &= 0 \\ \frac{\partial E(a,b,c)}{\partial c} &= 0 \end{aligned}$$

This may be expanded and then simplified to get the following system of equations in the three variables a, b, c.

$$\begin{array}{llllll} a(\sum\cos^2\theta_i) & +b(\sum\sin\theta_i\cos\theta_i) & +c(\sum\cos\theta_i) & = & \sum z_i\cos\theta_i \\ a(\sum\cos\theta_i\sin\theta_i) & +b(\sum\sin^2\theta_i) & +c(\sum\sin\theta_i) & = & \sum z_i\sin\theta_i \\ a(\sum\cos\theta_i) & +b(\sum\sin\theta_i) & +cN & = & \sum z_i \end{array}$$

Note that this matrix system is symmetric. The solution of this system gives us the optimal values of a, b, c.

1.3 Fitting a Plane to 3D data.

We are given the data $P = \{(x_1, y_1, z_1), \ldots, (x_N, y_N, z_N)\}$, and we would like to fit a plane to this data. A formulation of this is to write the plane as:

$$z = ax + by + c$$

The corresponding error function is:

$$E(a, b, c) = \sum (z_i - ax_i - by_i - c)^2$$

This results in the equations:

$$\begin{array}{llll} a(\sum x_i^2) & +b(\sum x_i y_i) & +c(\sum x_i) & = & \sum x_i z_i \\ a(\sum x_i y_i) & +b(\sum y_i^2) & +c(\sum y_i) & = & \sum y_i z_i \\ a(\sum x_i) & +b(\sum y_i) & +cN & = & \sum z_i \end{array}$$

These may easily be solved.

1.4 Fitting a Circle to 2D data.

We are given points $P = \{(x_1, y_1), \ldots, (x_N, y_N)\}$, and we would like to fit the best possible circle to this data. A naive attempt would be to let the proposed centre by (a, b) and the radius be r. We thus have $(x_i - a)^2 + (y_i - b)^2 - r^2$ as a measure of the error. However, note now that the error may be negative as well. In order to avoid this we may choose the error metric as:

$$E(a, b, r) = \sum_i [(x_i - a)^2 + (y_i - b)^2 - r^2]^2$$

However, we see at once that the derivatives of $E(a, b, r)$ with respect to its variables $a, b, r)$ are *cubic* and thus difficult to solve.

We give here another formulation. Let us suppose that we know an approximate centre and radius, say (A, B) and R. We translate the origin to (A, B) and convert the set P to a 3D set Q via a stereographic projection of radius $R/2$. This is illustrated in Figure 3.

For the point $p = (x, y)$ on the plane, we get the point

$$p' = (\frac{R^2}{x^2 + y^2 + R^2} x, \frac{R^2}{x^2 + y^2 + R^2} y, \frac{x^2 + y^2}{x^2 + y^2 + R^2} R)$$

We see that for p almost on the circle with radius R, z is close to $R/2$. Thus all the points $p \in P$ go close to the equator of the control sphere of the stereographic projection. Let us call these projected points as P'. We now use the earlier scheme to fit the best plane to P'. This plane interects the sphere in a circle C'. Project this circle back to the plan to get the best fitting circle C.

In other words, we form the intersection of the equations below to get C':

$$\begin{array}{rcl} x^2 + y^2 + (z - R/2)^2 - (R/2)^2 & = & 0 \\ ax + by + c - z & = & 0 \end{array}$$

Eliminating z, we get:

$$x^2 + y^2 + (ax + by + c - R/2)^2 - (R/2)^2 = 0$$

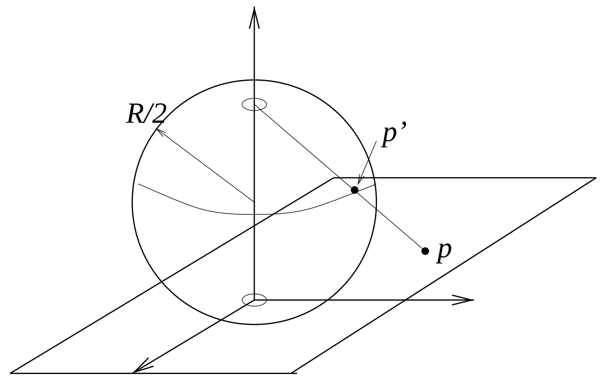

Figure 3: Stereographic Projection.

Using this, we compute three points $p'_i = (x'_i, y'_i, z'_i)$, for $i = 1, 2, 3$, on the circle C'. We use the stereographic projection to get points q_i, for $i = 1, 2, 3$, on the circle C. We may now compute the circle C as that which passes through q_1, q_2 and q_3. This is the required circle.

Thus the basic algorithm is as follows:

1. Input: array $P = \{(x_1, y_1), \ldots, (x_N, y_N)\}$ of points, approximate centre (A, B) and approximate radius R.

2. Let $S = \{(x_1 - A, y_1 - B), \ldots, (x_N - A, y_N - B)\}$.

3. For each point $p = (x, y)$ in S construct the point:

$$p' = (\frac{R^2}{x^2 + y^2 + R^2}x, \frac{R^2}{x^2 + y^2 + R^2}y, \frac{x^2 + y^2}{x^2 + y^2 + R^2}R)$$

   Let this collection be known as P'.

4. For P' as above, let $z = ax + by + c$ be the best fitting plane in 3D to P'.

5. Compute three points w_1, w_2, w_3 in 3D which lie on the intersection of the equations:

$$\begin{aligned} x^2 + y^2 + (z - R/2)^2 - (R/2)^2 &= 0 \\ ax + by + c - z &= 0 \end{aligned}$$

6. Map these points w_1, w_2, w_3 back to $\mathbb{R}^2$ as v_1, v_2, v_3 by the following equation. Given $x = (x, y, z)$ we obtain v as:

$$v = (\frac{R}{R - z}x, \frac{R}{R - z}y, 0)$$

7. Now translate v_1, v_2, v_3 by $(-A, -B)$ to obtain points v'_1, v'_2, v'_3. In other words, if $v = (x, y)$, then $v' = (x + A, y + B)$.

8. Finally obtain the best fit circle as the one passing through v'_1, v'_2, v'_3.

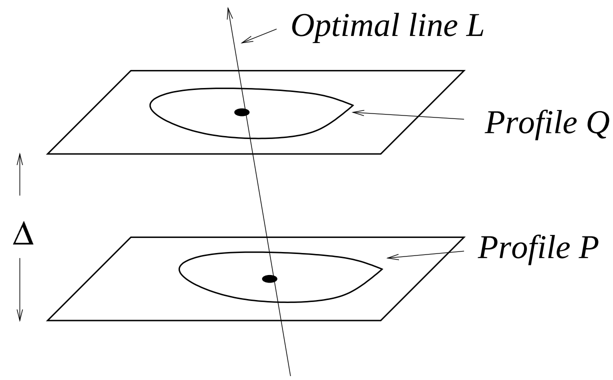

Figure 4: Optimal Axis.

1.5 Cylindricity Checking.

We are given, to begin with, two cross-sections of an extruded solid, at the heights d_1 and d_2. Furthermore, we assume that the profiles of the cross-sections are available as two arrays $Q = (q_1, \ldots, q_n)$ and $P = (p_1, \ldots, p_n)$, where $q_i = (x_i, y_i)$ and $p_j = (x'_j, y'_j)$. We assume that the point p_i and q_i are "vertically alligned", for all i.

We assume that the cylinder axis is defined by L (See Figure 4):

$$\begin{aligned} x &= az + b \\ y &= cz + d \end{aligned}$$

For this chosen axis, we compute the error:

$$E(a,b,c,d) = \sum_i [(x_i - ad_1 - b) - (x'_i - ad_2 - b)]^2 + [(y_i - cd_1 - d) - (y'_i - cd_2 - d)]^2$$

Let $\Delta = d_1 - d_2$. We have:

$$E(a,b,c,d) = \sum_i [x_i - x'_i - a\Delta]^2 + [y_i - y'_i - c\Delta]^2$$

We thus see that $E(a,b,c,d)$ just depends on a, c. Differentaiting with respect to a, c gives us that

$$\begin{aligned} a\Delta &= \frac{\sum_i (x_i - x'_i)}{n} \\ c\Delta &= \frac{\sum_i (y_i - y'_i)}{n} \end{aligned}$$

Having determined the slopes a and c, of the line L, we determine b and d using the lower of the two profile P. Let $X_P = (\sum_i x'_i)/n$ and $Y_P = (\sum_i y'_i)/n$. We solve for b and d using the equations:

$$\begin{aligned} X_P = ad_1 + b \\ Y_P = cd_1 + d \end{aligned}$$

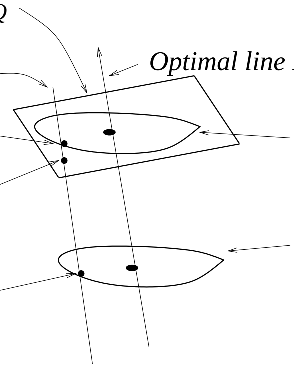

Figure 5: The correction.

Once this is done, we compute the point (X_Q, Y_Q), by

$$X_Q = ad_2 + b$$
$$Y_Q = cd_2 + d$$

We now construct the lines L_i as the one which passes through p_i and q_i, the i-th poinst on the profiles P and Q. We also construct the planes F_P and F_Q, which passes through (X_P, Y_P) (respectively (X_Q, Y_Q)), and is perpendicular to L. We now get the corrected profiles CP and CQ, where the i-th point on CP is given by cp_i as the intersection of L_i and F_P (See Figure 5).