

APPENDIX-B : System Capacity- Analysis and Optimisation

1 Spatial Reuse Model

Maximising the cardinality of independent sets used in a schedule need not necessarily increase the throughput, since as the cardinality of the set increases, the prevailing SINR drops, thereby resulting in an increase in the probability of error, decreasing the throughput. Hence it is necessary to limit the cardinality of the independent set used so as to satisfy the SINR requirements. i.e., there is a limit to the number of simultaneous transmissions possible.

In this section the problem of finding the maximum number of simultaneous transmissions possible in different sectors in the uplink and the downlink is being considered. There is no power control in the downlink. The BTS transmits to all the STs at the same power. There is static power control in the uplink. Each ST transmits to the BTS at a fixed power, such that the average power received from different STs at the BTS is the same. The STs near the BTS transmit at a lower power and the ones farther away transmit at a higher power.

A typical antenna pattern used in the deployment is as shown in Figure 1. Based on the antenna pattern, one can divide the region into an *association region*, a *taboo region* and a *limited interference region* with respect to each BTS.

The radial zone over which the directional gain of the antenna is above -3dB is called the association region. In the analysis, the directional gain is assumed to be constant over this region. Any ST which falls in this region of a BTS antenna j is associated to the BTS j .

The region on either side of the association region where the directional gain is between -3dB and -15dB is called the taboo region. Any ST in this region of BTS j causes significant interference to the transmissions occurring in Sector j . When a transmission is occurring in Sector j , no transmission is allowed in this region.

In the limited interference region the directional gain of the BTS antenna is below -15dB. A single transmission in this region of BTS j may not cause sufficient interference to the transmission in Sector j . But a number of such transmissions may add up causing the SINR of a transmission in Sector j to fall below the threshold required for error free transmission. This is taken care of by limiting the total number of simultaneous transmissions in the system as explained in Sections 1.1 and 1.2.

As an example, for the antenna pattern shown in Figure 1, the association region is a 60° sector centered at the 0° mark, the taboo region is 30° on either side of this association region, and the limited interference region covers the remaining 240° .

1.1 Uplink

In the uplink, there is static power control. All STs transmit at a power such that the power received at the BTS is P times noise power. Let the maximum power that can be transmitted by an ST be

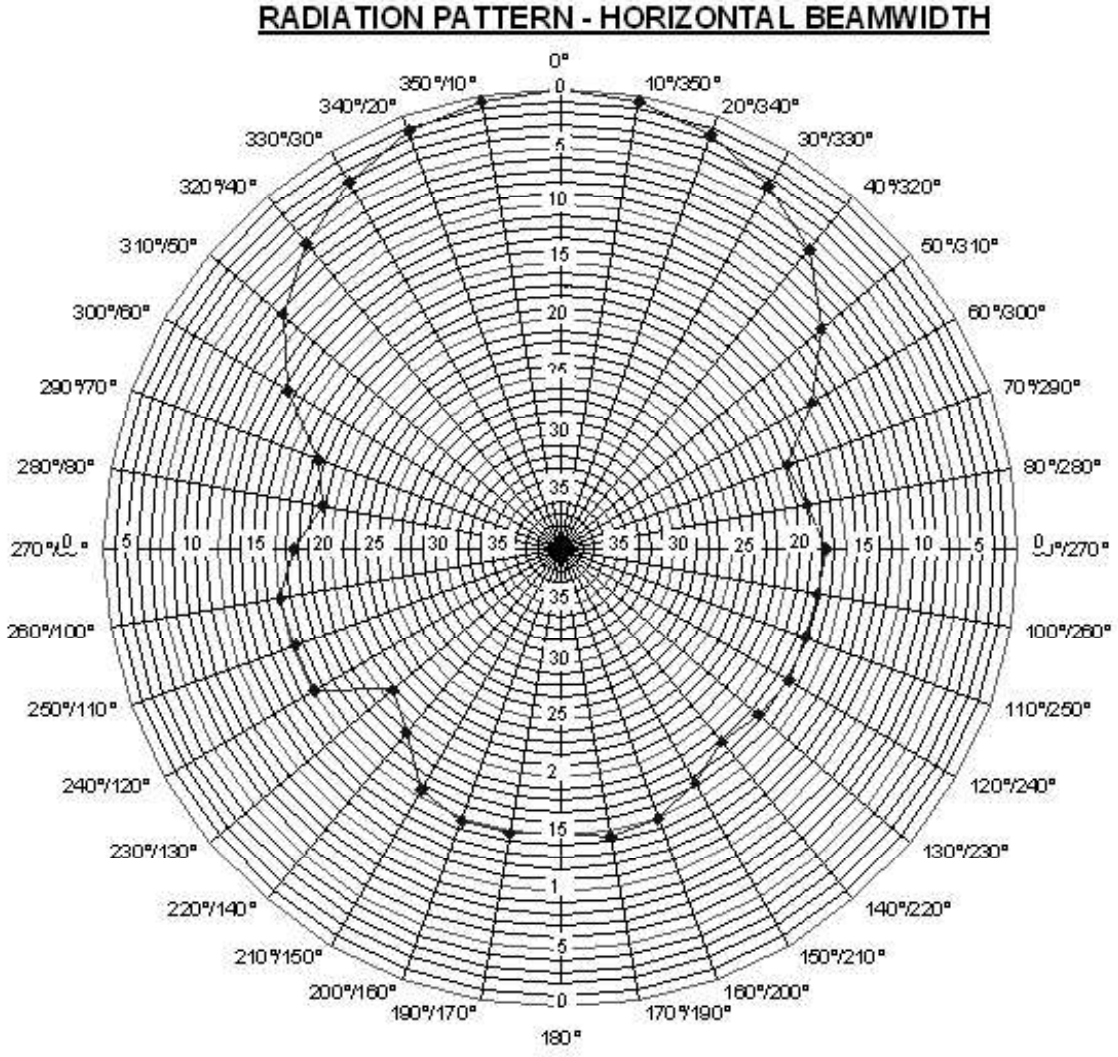


Figure 1: Radiation pattern for a typical antenna that could be used in the deployment.

P_t times noise power. Let R_0 be the distance such that when P_t is transmitted by an ST at distance R_0 , the average power received at the BTS is P_0 times noise power, where P_0 is the minimum SNR required to decode a frame with a desired probability of error. Also, let R be such that when P_t is transmitted from an ST at distance R , the power received at the BTS is P times noise power, i.e.,

$$\frac{P}{P_0} = \left(\frac{R}{R_0} \right)^{-\eta}$$

In the presence of interferers, the power required at the receiver will be greater than P_0 times noise. Let P be the power required, so that the receiver decodes the frame with a desired probability of error, in the presence of interferers. The directional gain of the BTS antenna is -15dB in the other non taboo directions. Hence, the interference power from a transmission in any other sector would be $10^{-\frac{3}{2}}P$. If there are $n_0 - 1$ simultaneous transmissions, the path loss factor being η , the signal

to interference ratio at the BTS receiver is

$$\begin{aligned}\Psi_{rcv} &= \frac{P}{1 + \sum_{i=1}^{n_0-1} 10^{-\frac{3}{2}} P} \\ &= \frac{P_0(\frac{R}{R_0})^{-\eta}}{1 + (n_0 - 1)10^{-\frac{3}{2}} P_0(\frac{R}{R_0})^{-\eta}}\end{aligned}$$

For decoding a frame with less than a given probability of error, we need an SINR of P_0 at the receiver. So, R should be such that

$$\begin{aligned}\Psi_{rcv} &\geq P_0 \\ \frac{P_0(\frac{R}{R_0})^{-\eta}}{1 + (n_0 - 1)10^{-\frac{3}{2}} P_0(\frac{R}{R_0})^{-\eta}} &\geq P_0 \\ n_0 &\leq 1 + \frac{(\frac{R}{R_0})^{-\eta} - 1}{10^{-\frac{3}{2}} P_0(\frac{R}{R_0})^{-\eta}}\end{aligned}$$

To provide a margin for fading, consider a reduced range R' such that

$$10 \log \left(\frac{R'}{R} \right)^{-\eta} \geq 2.3\sigma$$

where σ is the fade variance. In this case, 99% of the STs in a circle of radius R' around the BTS can have their transmit power set so that the average power P is received at the BTS in the uplink.

Evidently, n_0 can be increased by reducing R' . But then, spatial reuse increases at the expense of coverage. This tradeoff can be captured by the spatial capacity measure $C = nR'^2$, which has units slots.km² (or packets.km²)

The variation of the maximum number of transmissions, n_0 and system capacity, C , with coverage is as shown in Figure 2. One can see that for each η , there is an optimal n_0 and R' such that C is maximum. The coverage for which capacity is maximum can be obtained by equating the derivative of C , with respect to (R'/R_0) to be zero. Take $r' = \frac{R'}{R_0}$ and set

$$\frac{dC}{dr'} = 0$$

Then we get the optimum value of r' and n_0 as

$$\begin{aligned}r' &= \left(10^{-\frac{2.3\sigma}{10}} \frac{1 + 10^{-\frac{3}{2}} P_0}{1 + \frac{\eta}{2}} \right)^{\frac{1}{\eta}} \\ n_0 &= \frac{(1 + 10^{-\frac{3}{2}} P_0)\eta}{10^{-\frac{3}{2}} P_0(\eta + 2)}\end{aligned}$$

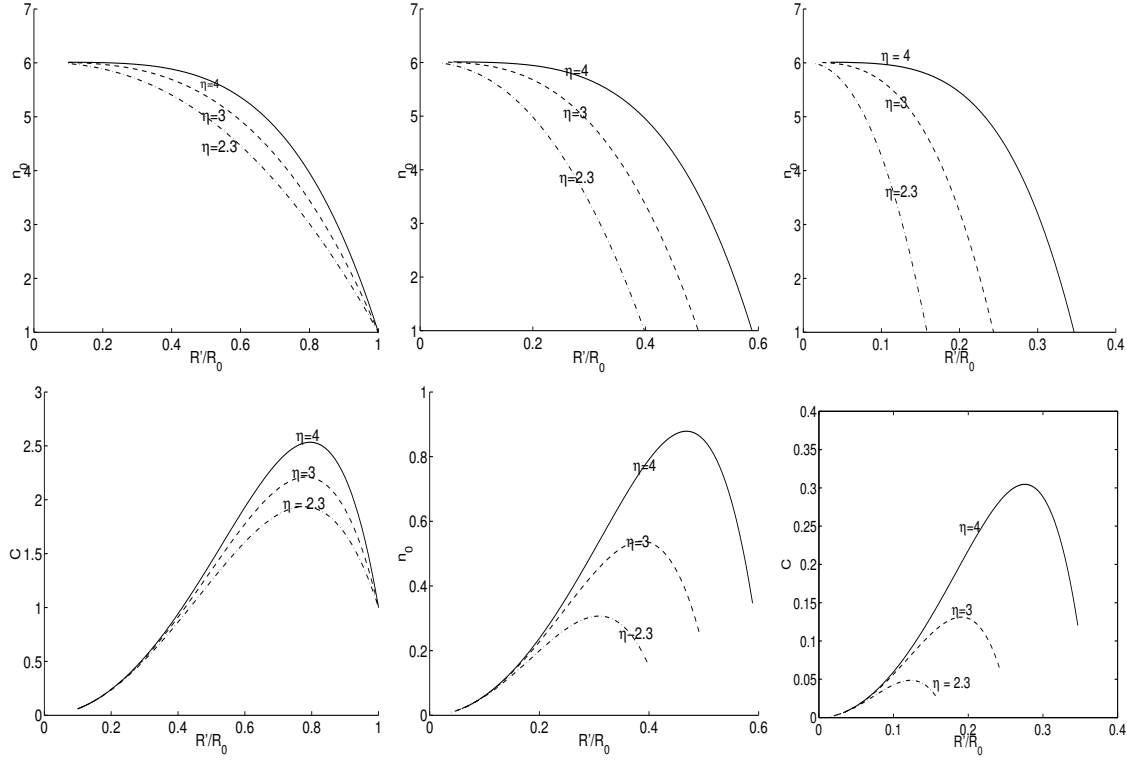


Figure 2: Variation of the number of simultaneous transmissions possible (n_0) and system capacity(C) with coverage relative to a reference distance R_0 for $\eta = 2.3, 3, 4$ and $\sigma = 0, 4, 8$. Plots for $\sigma = 0, 4, 8$ are shown left to right.

σ η	0	4	8
2.3	0.77	0.31	0.12
3	0.78	0.39	0.20
4	0.80	0.47	0.28

η	2.3	3	4
n_0	3	3	4

Table 1: The optimum values of C and n_0 for different values of η and σ .

Table 1 gives the optimum coverage C and maximum number of simultaneous transmissions possible for different values of η and σ . P_0 is taken to be $8dB$. Directional gain of the antenna is taken to be 1 in the associated sector and $-15dB$ in non-taboo directions. For path loss factor $\eta = 4$, the number of simultaneous transmissions is seen to be 4. For a given value of η , the maximum number of simultaneous transmissions is found to be independent of the fade variance σ .

1.2 Downlink

In the downlink, the transmit power is kept constant. The BTS antennas transmit at a power P_t times noise. Let R_0 be the distance at which the average power received is P_0 times noise. R be the distance such that the average power received is P . Then,

$$\begin{aligned} P_0 &= P_t \left(\frac{R_0}{d_0} \right)^{-\eta} \\ P &= P_t \left(\frac{R}{d_0} \right)^{-\eta} \\ \frac{P}{P_0} &= \left(\frac{R}{R_0} \right)^{-\eta} \end{aligned}$$

Allowing $n_0 - 1$ interferers,

$$\begin{aligned} \Psi_{rcv} &= \frac{P}{1 + (n_0 - 1)10^{-\frac{3}{2}}P} \\ &= \frac{P_0 \left(\frac{R}{R_0} \right)^{-\eta}}{1 + (n_0 - 1)10^{-\frac{3}{2}}P_0 \left(\frac{R}{R_0} \right)^{-\eta}} \end{aligned}$$

which is the same as in uplink. So, the optimum number of transmissions and optimum coverage in uplink and downlink are the same. The plots and tables for uplink apply for downlink also.

1.3 Number of Sectors

Once the maximum number of simultaneous transmissions possible, n_0 is obtained, one gets some idea about the number of sectors required in the system. In an n_0 sector system, when a transmission occur in the taboo region between Sector j and Sector $j + 1$, no more transmissions can occur in Sectors j and $j + 1$. So, the number of simultaneous transmissions can be at most $n_0 - 1$, one in Sector j and $j + 1$ and at most one each in each of the other sectors. Thus the maximum system capacity cannot be attained with $n_0 - 1$ sectors. With $n_0 + 1$ sectors, one can choose maximal independent sets such that the sets are of cardinality n_0 . So, at least $n_0 + 1$ sectors are needed in the system. From the spatial reuse model it can be seen that there can be up to 4 simultaneous transmissions in the system, for path loss $\eta = 4$. So, the system should have at least 5 sectors.

2 Characterising the Average Rate region

There are m STs. Suppose a scheduling policy assigns $k_j(t)$ slots, out of t slots, to ST j , such that $\lim_{t \rightarrow \infty} \frac{k_j(t)}{t}$ exists and is denoted by r_j . Let $\mathbf{r} = (r_1, r_2, \dots, r_m)$ be the rate vector so obtained. Denote by $\mathcal{R}(n)$ the set of achievable rates when the maximum number of simultaneous transmission permitted is n . Notice that for $n_1 > n_2$, $\mathcal{R}_1 \supset \mathcal{R}_2$. This is evident because any sequence of scheduled slots with $n = n_2$ is also schedulable with $n = n_1$. In the previous section, we have determined the maximum value of n , i.e., n_0 . Denote $\mathcal{R}_0 = \mathcal{R}(n_0)$. A scheduling policy will achieve an $\mathbf{r} \in \mathcal{R}_0$. In this section, we provide some understanding of \mathcal{R}_0 via bounds.

2.1 An Upper Bound on Capacity

Suppose each ST has to be assigned the same rate r . In this subsection an upperbound on r is determined. In general, the rate vector $(r, r, \dots, r) \notin \mathcal{R}_0$. The upper bound is obtained via simple linear inequalities. Consider the case $n \geq 3$. Suppose one wishes to assign an equal number of slots k to each ST in the uplink. There are N_U uplink slots in a frame. Consider Sector j , which contains m_j STs. Thus $k \cdot m_j$ slots need to be allocated to uplink transmission in Sector j . When STs in the interference region $j-$ or $j+$ transmit, then no ST in Sector j can transmit. Suppose $k_{j\pm}$ slots are occupied by such interference transmission. Now it is clear that

$$k \cdot m_j + k_{j\pm} = N_U$$

because whenever there is no transmission from the interference region for sector j there can be a transmission from sector j . Let m_{j-} and m_{j+} denote the number of STs in the interference regions adjacent to Sector j . Since the nodes in $j-$ and $j+$ can transmit together, we observe that

$$k_{j\pm} \geq \max(k \cdot m_{j-}, k \cdot m_{j+})$$

with equality if transmission in $j-$ and $j+$ overlap wherever possible. Hence one can conclude that for any feasible scheduler that assigns k slots to each ST in the uplink

$$k \cdot m_j + \max(k \cdot m_{j-}, k \cdot m_{j+}) \leq N_U$$

For large frame time N , divide the above inequality by N and denote the rate of allocation of slots by r . Thus if out of t slots, each ST is allocated k slots, then $r = \lim_{t \rightarrow \infty} \frac{k}{t} \leq 1$

$$r \cdot m_j + r \cdot \max(m_{j-}, m_{j+}) \leq \phi_u$$

where ϕ_u is the fraction of frame time allocated to the uplink or

$$r \leq \frac{\phi_u}{m_j + \max(m_{j-}, m_{j+})}$$

This is true for each j . So,

$$r \leq \frac{\phi_u}{\max_{1 \leq j \leq n} (m_j + \max(m_{j-}, m_{j+}))}$$

For the case $n = 2$ for $j \in \{1, 2\}$ denote the interfering nodes in the other sector by m_j . One easily sees that

$$r \leq \frac{\phi_u}{\max(m_1 + m'_1, m_2 + m'_2)}$$

2.2 An Inner Bound for the Rate Region

In this section a rate set \mathcal{R}_L is obtained such that $\mathcal{R}_L \subset \mathcal{R}_r$. i.e., \mathcal{R}_L is an inner bound to the achievable rate set.

The following development needs some graph definitions.

Reuse constraint graph: Vertices represents links. In any slot all links are viewed as uplinks or all are downlinks. Two vertices in the graph are connected, if a transmission in one link can cause interference to a transmission in the other link. The reuse constraint graph is represented as $(\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of vertices and \mathcal{E} is the set of edges.

Clique: A fully connected subgraph of the reuse constraint graph. A transmission occurring from an ST in a clique can interfere with all other STs in the clique. At most one transmission can occur in a clique at a time.

Maximal clique: A maximal clique is a clique which is not a proper subgraph of another clique.

Clique incidence matrix: Let κ be the number of maximal cliques in $(\mathcal{V}, \mathcal{E})$. Consider the $\kappa \times m$ matrix \mathcal{Q} with

$$\mathcal{Q}_{i,j} = \begin{cases} 1 & \text{if link } j \text{ is in clique } i \\ 0 & \text{o.w.} \end{cases}$$

By the definition of \mathbf{r} and \mathcal{Q} , a necessary condition for \mathbf{r} to be feasible is (denoting by $\mathbf{1}$, the column vector of all 1s.)

$$\mathcal{Q} \cdot \mathbf{r} \leq \mathbf{1}$$

since at most one link from a clique can be activated. In general, $\mathcal{Q} \cdot \mathbf{r} \leq \mathbf{1}$ is not sufficient to guarantee the feasibility of \mathbf{r} . It is sufficient if the graph is linear. A linear graph is one in which links in each clique is contiguous. A linear clique will have a clique incidence matrix of the form

$$\mathcal{Q} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \end{bmatrix}$$

The reuse constraint graph in the multisector scheduling problem being considered has a ring structure. $\mathcal{Q} \cdot \mathbf{r} \leq 1$ gives an upper bound on the rate vector. The reuse constraint graph is linear except for the wrapping around at the end. If the nodes in one sector are deleted, the graph becomes linear. Let \mathbf{m}_i be the set of STs in Sector i . There is a feasible \mathbf{r}_i such that $\mathcal{Q} \cdot \mathbf{r}_i \leq 1$ and all STs in \mathbf{m}_i are given rate 0. Linear combination of feasible vectors is also feasible. Thus, defining

$$\mathcal{R}_L := \left\{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^m \alpha_i \mathbf{r}_{ij}; \quad \mathcal{Q} \cdot [\mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_m] \leq \mathbf{1}; \quad \sum_{i=1}^m \alpha_i = 1; \quad \mathbf{r}_1(\mathbf{m}_1) = \dots = \mathbf{r}_m(\mathbf{m}_m) = 0 \right\}$$

We see that $\mathcal{R}_L \in \mathcal{R}$

2.3 Optimum Angular positioning of the Antennas

As can be seen from the previous section, feasible rates set, \mathcal{R}_0 , of the system depends on the spatial distribution of the STs around the BTS. Thus the \mathcal{R}_0 varies as the sector orientation is changed. A system where the antennas are oriented in such a way that most STs fall in the association region of BTSs rather than in the taboo region will have more capacity than one in which more STs are in the taboo regions.

One sector boundary is viewed as a reference. Let $\mathcal{R}_0(\theta)$ denote the feasible rate set, when this boundary is at an angle θ with respect to a reference direction. Then, for each $0 \leq \theta \leq \frac{360^\circ}{n}$, we have $\mathcal{R}_0(\theta)$, where n is the number of sectors. Since $\mathcal{R}_0(\theta)$ is not known, the inner bound $\mathcal{R}_L(\theta)$ is used in the following analysis. If each vector \mathbf{r} is assigned a utility function $U(\mathbf{r})$, then one could seek to solve the problem

$$\max_{0 \leq \theta \leq \frac{360^\circ}{n}} \max_{\mathbf{r} \in \mathcal{R}_L(\theta)} U(\mathbf{r})$$

and then position the antenna at this value of θ .

The optimization can be done so as to maximise the average rate allocated to each ST, with the constraint that each ST gets the same average rate. The bound evaluated with average rate to each ST, for antenna positions differing by 5° is given below. $ub(i)$ gives the upperbound on capacity of the system with antenna placed at $((i-1) * 5)^\circ$ from the reference line. Similarly $lb(i)$ is the lower bound for each position.

$$\text{Upper bound, } \mathbf{ub} = \begin{bmatrix} 0.0714 & 0.0769 & 0.0714 & 0.0714 & 0.0667 & 0.0769 & 0.0714 \\ 0.0667 & 0.0667 & 0.0625 & 0.0625 & 0.0667 & 0.0667 & 0.0769 \end{bmatrix}$$

$$\text{Lower bound, } \mathbf{lb} = \begin{bmatrix} 0.0714 & 0.0769 & 0.0714 & 0.0714 & 0.0667 & 0.0769 & 0.0714 \\ 0.0667 & 0.0667 & 0.0625 & 0.0625 & 0.0667 & 0.0667 & 0.0769 \end{bmatrix}$$

The bounds are seen to be very tight, and the maximum rate is obtained when antennas are at 5° , 25° or -5° from the reference line. The maximum rate so obtained is 0.0769, giving a sum capacity of 3.076. Only 14 different positions of the antenna are considered for a 5 sector system, since the pattern would repeat itself after that.

Trying to optimize the rates such that the rate to each ST is maximized will adversely affect the sum capacity of the system. So, take $U(\mathbf{r}) = \sum_{j=1}^m (r_j)$.

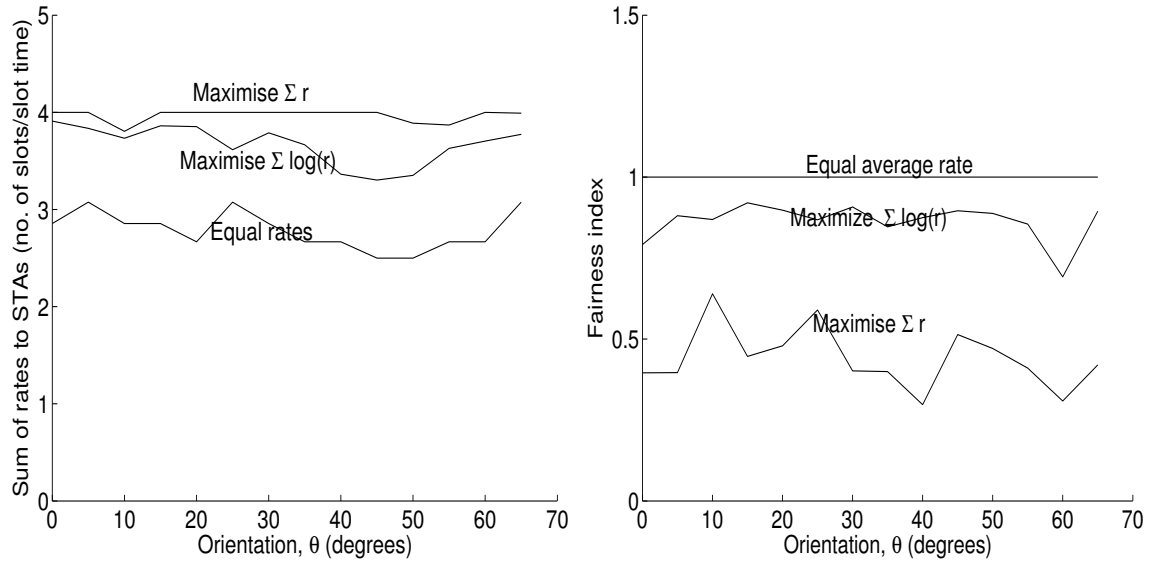


Figure 3: Variation of sum rate and fairness index with antenna orientation for different utility functions.

For example, the sum capacity evaluated for antenna position varied in steps of 5° is as follows. It can be seen that the system capacity does not vary much with the position of the antenna. But, there seems to be some positions which are worse than the others.

Upper bound, $\mathbf{ub} = [4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4]$

Lower bound, $\mathbf{lb} = [4 \ 4 \ 3.8046 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 3.8883 \ 3.8699 \ 4 \ 3.9916]$

Maximising $\sum_{i=1}^m \log(r(i))$ under the given constraints for upper bound and lower bound gives the utility functions for different positions of the antenna as

$$U_{lb} = \begin{bmatrix} -96.2916 & -95.7459 & -97.0998 & -95.112 & -95.9083 & -98.0191 & -96.1752 \\ -99.1991 & -101.1465 & -102.5186 & -102.0872 & -99.52350 & -98.37440 & -96.58240 \end{bmatrix}$$

$$U_{ub} = \begin{bmatrix} -96.2916 & -95.5485 & -97.0998 & -95.1122 & -95.9083 & -98.0191 & -96.1752 \\ -99.1991 & -101.1465 & -102.5186 & -102.0872 & -99.5235 & -97.3909 & -96.4809 \end{bmatrix}$$

The sum capacities for each of the rates above are bounded by

$$\text{Sum of rates for upperbound} = \begin{bmatrix} 3.9102 & 3.8463 & 3.7333 & 3.8611 & 3.8535 & 3.6159 & 3.7896 \\ 3.6663 & 3.3637 & 3.30270 & 3.3513 & 3.6291 & 3.7565 & 3.7844 \end{bmatrix}$$

$$\text{Sum of rates for lower bound} = \begin{bmatrix} 3.9102 & 3.8350 & 3.7333 & 3.8612 & 3.8535 & 3.6157 & 3.7896 \\ 3.6663 & 3.3636 & 3.3027 & 3.3512 & 3.6291 & 3.704 & 3.7734 \end{bmatrix}$$

The utility function is maximum when the antenna is positioned at 15° from the reference line. The sum of rates at this position is 3.86. This gives a trade-off between maximising the system capacity and providing fairness.

The sum of the rate given to STs and the fairness index vs antenna orientation is plotted in Figure 3 for different utility functions (the lower bounds are plotted here). Fairness index varies from 0 to

1. For a rate vector \mathbf{r} , the fairness index is given by

$$\gamma = \frac{(\sum_{i=1}^m x_i)^2}{m \sum_{i=1}^m x_i^2}$$

. If the rates to different STs are equal, then fairness index would be 1, and it decreases as the rates are made unfair. The plots for maximum $\sum_{i=1}^m r_i$, maximum $\sum_{i=1}^m \log r_i$ and by maximising the average rate to each ST are shown. It can be seen that maximising the sum rate gives high overall capacity, but poor fairness. On the other hand, maximising the average rate to each ST gives good fairness, but low sum capacity. Maximising $\sum_{i=1}^m \log r_i$ gives a good tradeoff between maximising the system capacity and providing fairness. It is interesting to note that in maximum $\sum_{i=1}^m \log r_i$ case, the sum capacity is higher when fairness is lower and viceversa. For example, at $\theta = 10$, we can see that the sum rate is close to 4. The fairness index is also close to 1. So, we may choose this orientation as optimum