1. Show that a connected acyclic graph has $n-1$ edges.
   Notation: In general, n will denote the number of vertices and m, the number of edges.

2. Show that in a graph with every vertex degree at least 2 there must be a cycle.

   Hint. Consider a longest path in the graph.

3. A graph is *2-connected* or *2-vertex connected* if the removal of any vertex leaves it connected. Show that in a 2-connected graph every pair of vertices lie on a cycle.

   Hint. Induction on the distance between the two vertices.

4. A *tournament* is a directed graph such that for every pair of vertices u, v either there is an edge from u to v or from v to u. Show that a tournament has a directed path of length $n-1$.

5. **Mantel's Theorem**. Prove that the number of edges in a triangle free graph is at most $n^2/4$.

   Hint. Let $\{u, v\}$ be an edge. Remove these two vertices and induct on $V(H) = V(G) - \{u, v\}$. What is the base case?

6. **Turan's Theorem.** Prove that the number of edges in a K_{r+1}-free graph is at most $(1-1/r)(n^2/2)$.

   Hint. Induction on the pair (r, n). Generalise the previous theorem.

7. **Euler's Formula** A *planar* graph is a graph that can be drawn on the plane. The plane then is divided into regions or faces. Intuitively you can go from any point in a face to any other point without crossing any edge of vertex of the graph. Each face is bounded by a cycle. If G is a connected planar graph show that $n - m + f = 2$ where f is the number of faces.

   Hint. Induction on f. What is the base case?

8. **Hall's theorem:** Given a bipartite graph prove that it has a perfect matching if and only if for every subset S of A, the size of the neighborhood of S is at least that of S.

9. Prove that a graph with 6 vertices contains either an independent set or a clique of size 3.

10. **Graph Ramsey Theorem.** Prove that for every positive integers k, r there is a positive integer $N(k, r)$ such that every graph on $N(k, r)$ vertices either has a clique of size k or an independent set of size r.

    Hint. Induction on $k + r$. Take any vertex and examine its neighborhood. Write a recurrence for $N(k, r)$.

11. Prove that for a bipartite graph, the size of a maximum matching equals the size of the minimum vertex cover. First prove that if this is so, then the vertex cover has to contain exactly one end-point from each edge in the matching.

12. A coloring of a graph is an assignment of positive integers to the vertices of the graph such that the end-points of the edges are assigned different colors. Prove that if the maximum degree of a vertex in a graph is Δ then the graph can be colored using $\Delta + 1$ colors.

    Optional. For which graphs do you need $\Delta + 1$ colors (and not less)?