Lecture 1: Introduction

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Consider the following problem.

$$\max sinx$$

When the domain of x is not specified, we assume it is the set of all reals. The answer to this is one. This is an example of a maximization problem. If we had a min instead the answer would be -1. These are examples of unconstrained optimization problems where we wish to determine the maximum/ minimum of a function defined over the set of all real values of the argument. Typically however, we have a range of x for which we wish to calculate the function. Say,

$$\max sinx$$

$$-.02 \leq x \leq .06$$

Here too we can determine the maximum value. This is an example of constrained optimization of a function of one variable. We will typically have functions on many variables; however our focus will be on very special kinds of functions. The restrictions on the range will also be special.

The subject of Linear Optimization or Linear Programming first came up for study in the manufacturing and industrial engineering areas in the 1940s.

We begin with an example.

A factory needs to produce n different goods from m different kinds of raw materials. Each good, in the process of its manufacture, uses a certain proportion of each available raw material. The amount of each raw material available is limited. Each good manufactured yields a certain amount of profit.

The problem is to determine how much quantity of each good to produce using these raw materials in order to maximize the total profit on the goods.

We will allow fractional amounts of goods to be produced.

You are given:

- Raw Materials Available: $M_1, M_2, M_3, \cdots, M_m$
- Quantity of each material available: $b_1, b_2, b_3, \cdots, b_m$
- Goods to be produced: $G_1, G_2, G_3, \cdots, G_n$
- Profit per unit of each good: $p_1, p_2, p_3, \cdots, p_n$
- A_{ij}: Proportion of material M_i required to produce one unit of good G_j

Optimization Problem: How much of each good to produce, to maximize the total profit, under the given constraints?

This is a problem in algorithm design. The input is A_{ij} for each i and j, b_j for each j and c_i for each i. As output we desire the amount of each good to produce.

From early childhood we are told to let an unknown quantity be x. We continue this tradition. Let x_j denote the amount of good j that we should produce. Then the x_js must satisfy the following constraints. For each i, $A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n \leq b_i$. And, $x_j \geq 0$, for each j.

Subject to these constraints we wish to maximise $p_1x_1 + p_2x_2 + \cdots + p_nx_n$.

Or more succinctly:

$$
\begin{array}{ccc}
& \max c^T x & \\
& \text{such that} & \\
Ax & \leq & b \\
x & \geq & 0
\end{array}
$$

Note that c, x are n-dimensional vectors, b is m-dimensional and A is a $m \times n$ matrix.

Let us restate the problem. Among all vectors x that satisfy $Ax \leq b$ and $x \geq 0$ find one which maximises the quantity $c^T x$.

The name *Linear Optimization* comes from the fact that the quantity which is to be optimized is a linear function (we will define this later) of the unknown vector x and the constraints on x_j are linear inequalities.

Unlike some of the discrete algorithmic problems that we have encountered so far, in this problem, the number of xs which satisfy $x \geq 0$ and $Ax \leq b$ may be infinite. So even a brute force algorithm that terminates with the correct answer is not obvious. This will be our first goal.

To do so we need to understand what the set $\{x : Ax \leq b\}$ looks like. We begin by considering the simpler object $\{x : Ax = b\}$. Note that this may be an infinite set: the set of all solutions to a collection of m equalities.

Suppose there were only two variables x_1 and x_2. Linear equations of the form $a_{i1}x_1 + a_{i2}x_2 = b_i$ represents a set of straight lines in two dimensions with x_1 and x_2 as co-ordinates.

The solution to these equations can be a single point in the plane or a line or the empty set. The important point is it cannot look like anything else. We will show this formally soon. This can be extended further when there are 3 or more variables. Our goal is to have a geometric interpretation and an algebraic description for these sets.