Lecture 10: $Ax \le b$ as a convex combination of its extreme points

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In this lecture, we complete the proof of a theorem stating that all points in the set $Ax \le \mathbf{b}$ can be expressed as a convex combination of its extreme points. We then prove that a linear function on such a set is maximized at an extreme point. This leads us to certain algorithms for linear programming.

THEOREM 1 *Let $p_1, p_2, p_3, \ldots, p_t$ be the extreme points of the convex set $S = \{x : Ax \le \mathbf{b}\}$ Then every point in S can be represented as $\sum_{i=1}^{t} \lambda_i p_i$, where $\sum_{i=1}^{t} \lambda_i = 1$ and $0 \le \lambda_i \le 1$*

PROOF: The proof is by induction on the dimension of the object $\{x \ : \ Ax \le \mathbf{b}\}$. The base step is when the dimension is zero and is trivial.

For the inductive step consider an object $\{x \ : \ Ax \le \mathbf{b}\}$ in n-dimensions. Consider $p \in S$. For simplicity of presentation, assume that every inequality is strict for p. That is $Ap < \mathbf{b}$. Join p_1 to p and extend this line to meet a point q on the boundary of $\{x \ : \ Ax \le \mathbf{b}\}$. Note that the segment joining p_1 and p must lie inside the set by convexity. Also, such a point q must exist since the object is bounded. What does it mean that q is a boundary point? As we said before, it means that if we draw a small sphere with q as the center, then part of the sphere will be outside the set. Algebraically, this means that some inequality must be tight (must become an equality) at q. Recall the examples. It is likely that more than one inequality could become equalities. For simplicity, we will assume that exactly one, in fact the first one, becomes an equality.

For the point q, we must then have,

$$A_1 q = b_1 \tag{1}$$
$$A^{''} q < \mathbf{b}^{''} \tag{2}$$

$A^{''}$ is the rest of A. The object above is the intersection of two sets: $\{x : A_1 x = b_1\}$ and $\{x : A^{''} x < \mathbf{b}^{''}$. What does it mean to use induction on dimension? The first object has dimension $n-1$. So intersection with any other object will yield an object of smaller dimension and we can recurse. There are two issues to be sorted out. One is that the inductive assertion only works for objects of dimension $n-1$, of the type $\{x \ : \ Ax \le \mathbf{b}\}$. Why is the intersection of the two sets mentioned above of this type?

The trick is to use the first equality and solve for one variable, and replace it throughout in $A^{''}$. Which variable? Can we use any variable? Not exactly. We can remove a variable i if A_{1i} is non-zero. We then solve for this variable to get $x_i = \frac{1}{A_{1i}}(b_1 - \Sigma_{j=2}^{n} A_{1j} x_j)$. We then substitute this in every inequality to get a new set of inequalities with one less variable.

This is a new convex set $S^{'} = \{x : Cx \le \mathbf{d}\}$, in *one less dimension*. This convex set is the intersection of two convex sets $A_1 x = b_1$ and $\{x \ : \ Ax \le \mathbf{b}\}$. Why is this?

Exercise: Show that any point in one set must be contained in the other. Also show a bijection between the two sets.

By the induction hypothesis, q can be written as a convex combination of extreme points in this object, $S^{'}$. Hence,

$$p = \beta p_1 + (1-\beta) q \tag{3}$$
$$= \beta p_1 + (1-\beta) \sum_{i=1}^{t^{'}} \gamma_i q_i \tag{4}$$

Strictly, q is a point in the old object. It has a mirror image in the new object. But we will talk of the two as the same.

This convex combination is, however, in terms of the extreme points $q_1, q_2, q_3, \ldots, q_{t'}$ of $S^{'}$. We need to show that the extreme points of $S^{'}$ are also extreme points of S. Suppose they were not. Let $p^{'}$ be an extreme point of $S^{'}$ but not of S. Then $\exists\ p_1^{'}, p_2^{'} \in S$ such that $p^{'} = \lambda p_1^{'} + (1-\lambda)p_2^{'}$. Hence:

$$\begin{aligned} b_1 &= A_1 p^{'} & (5)\\ &= \lambda A_1 p_1^{'} + (1-\lambda) A_1 p_2^{'} & (6)\\ &\leq \lambda b_1 + (1-\lambda) b_1 = b_1. & (7) \end{aligned}$$

The first equality is because p' is in S', and the inequality is because both p_1 and p_2 are in S. Which means the inequality must be an equality. Therefore $p_1^{'}$ and $p_2^{'}$ must also be in $S^{'}$. Hence the point $p^{'}$ cannot then be extreme in $S^{'}$, as it is the convex combination of two points in the same set.

This completes the proof. □ The following theorem will put the last detail in place to enable construction of an algorithm for solving LP problems.

THEOREM 2 *A linear function on $S = \{x : Ax \leq \mathbf{b}\}$ is maximized at an extreme point.*

PROOF: Notice that we have tackled something like this before. Taking a convex combination is like taking a weighted average of points. The previous theorem essentially says that every point is the weighted average of the extreme points. If we consider linear functions, the value at any point is the weighted average of the values at extreme points. The result follows. Let us formalise this.

Let a linear function f attain its maximum at point p, where $p = \sum_{i=1}^{t} \lambda_i p_i$. Then $f(p) = \sum_{i=1}^{t} \lambda_i f(p_i)$. If all of the $f(p_i)$'s were smaller than $f(p)$, their weighted average cannot sum to $f(p)$. Therefore for at least one i, $f(p_i) = f(p)$. □

Having proved this, we have a finite algorithm at our disposal now. An extreme point is an intersection of n linearly independent hyperplanes. We examine all combinations of n rows from A ($\binom{m}{n}$ of them), solve for x_0 in $A^{'} x_0 = \mathbf{b}'$ using Gaussian Elimination, **verify** that the solution indeed satisfies all other inequalities, and then calculate $c^T x$.

The verification part is important, as the n hyperplanes we choose may end up defining an infeasible point. An example is 2-D is shown in Figure 1.

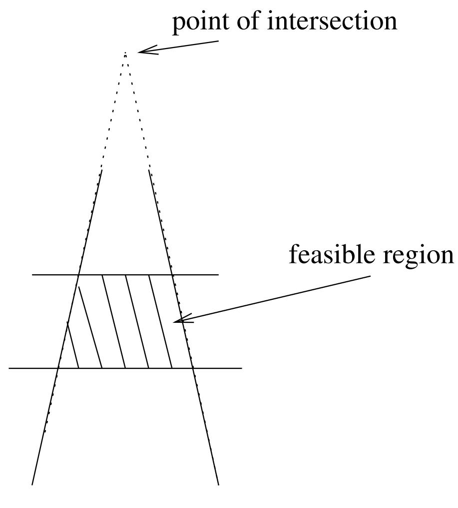

Figure 1: Why we need to verify