Lecture 15: Complementary Slackness

Lecturer: *Sundar Vishwanathan*
Computer Science & Engineering | Indian Institute of Technology, Bombay

1 Complementary Slackness

Primal	Dual
$\max c^T x$	$\min y^T b$
$Ax \leq b$	$A^T y = c$
	$y \geq 0$

We proved this in the last lecture.

Theorem 1 *If the primal is feasible and the cost is bounded, then the dual is feasible and its cost is also bounded. Moreover, their optimum values coincide.*

The following theorem follows from the above theorem.

Theorem 2 *Consider an x_0 and y_0, feasible in the primal and dual respectively. Then both are optimum iff $\mathbf{c}^T x_0 = y_0^T \mathbf{b}$.*

The proof is an exercise. One way is to use the hint given below. Note that $\mathbf{c}^T x_f = y_f^T Ax \leq y_f^T \mathbf{b}$ for any feasible x_f and y_f.

Theorem 3 (Complementary Slackness) *Consider an x_0 and y_0, feasible in the primal and dual respectively. That is, $Ax_0 \leq \mathbf{b}$ and $A^T y_0 = \mathbf{c}$; $y_0 \geq \mathbf{0}$. Then $\mathbf{c}^T x_0 = y_0^T \mathbf{b}$ if and only if $(y_0)_i > 0 \Rightarrow A_i x_0 = \mathbf{b}_i$.*

Note that this gives a new criterion for optimality. The second criterion is called *Complementary Slackness.* It says that if a dual variable is greater than zero (slack) then the corresponding primal constraint must be an equality (tight.) It also says that if the primal constraint is slack then the corresponding dual variable is tight (or zero.)

Notice that if y_0 were an extreme point in the dual, the complementary slackness condition relates a dual solution y_0 to a point x_0 in the set F in the primal. When we add to this, the fact that x_0 is feasible, we may infer that both points should be optimal.

We prove this formally below.

Proof: First assume that the complementary slackness condition holds. We need to prove that the costs of the points are equal.

$$\begin{aligned}
y_0^T b &= \sum_{j=1}^{m} y_{0j} b_j \\
&= \sum_{j=1}^{m} y_{0j}(A_j x_0) \qquad \text{(using complementary slackness)} \\
&= \sum_{j=1}^{m} y_{0j} \left(\sum_{i=1}^{n} A_{ji} x_{0i} \right) \\
&= \sum_{i=1}^{n} x_{0i} \left(\sum_{j=1}^{m} A_{ji} y_{0j} \right) \\
&= x_0^T c \qquad \text{(using } A^T y_0 = c \text{)} \\
&= c^T x_0
\end{aligned}$$

Now assume that the costs are equal.

$$\begin{aligned}
c^T x_0 &= x_0^T c \\
&= \sum_{i=1}^{n} x_{0i} \left(\sum_{j=1}^{m} A_{ji} y_{0j} \right) \\
&= \sum_{j=1}^{m} y_{0j} \left(\sum_{i=1}^{n} A_{ji} x_{0i} \right) \\
&= \sum_{j=1}^{m} y_{0j}(A_j x_0) \\
&\leq \sum_{j=1}^{m} y_{0j} b_j \qquad \text{(using } y_{0j} \geq 0 \text{ and } A_j x_0 \leq b \text{)} \\
&= y_0^T b
\end{aligned}$$

But we know that $c^T x_0 = y_0^T b$. Hence, $y_0 > 0 \Rightarrow A_i x_0 = b_i$. □

2 Infeasibility and Unboundedness

Every LP problem is either feasible or infeasible. Infeasibility means that there is no solution to $\{x \ : \ Ax \leq \mathbf{b}\}$. *Exercise:* Give an example of an infeasible LP.

Feasible problems either have a solution or are unbounded. This will depend on the cost function. *Exercise:* Give an example of a feasible but unbounded LP.

What happens to the dual in these cases? In particular can both the primal and dual be infeasible? The answer is yes.

Exercise. Exhibit such a primal-dual pair.

In the ensuing discussion we will assume that at least one of the primal and dual is feasible. We first note that if an LP is unbounded then the dual will be infeasible. If the dual were feasible then the cost at any feasible point of the dual is an upper-bound on the cost of the optimal primal solution. Prove this.

Exercise: If the LP is infeasible, and the dual is feasible prove that the dual will be unbounded.

We will collect all this in the form of a theorem in the next lecture.

A Point on Degeneracy: In general, LPs need not be degenerate. Our description of Simplex, its proof of correctness and the subsequent proof of the Duality theorem uses this assumption. However all of this holds even when the LPs are not degenerate. The description of Simplex needs to be modified though. An additional problem that arises then is cycling. We will not bother about these points in this course however we will assume the theorems in general. The fact that at an optimum vertex the cost vector can be written as a non-negative linear combination of the normals is also true when the optimal point is degenerate. These are good exercises to try. Some hint will be given in the next lecture.