Lecture 16: Duality: Bounding the Primal Solution from Below

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In this lecture we will see another motivation for duality. However, before that, a quick recap.

Consider the following primal-dual pair.

$$\begin{array}{llll} \text{P}: & \max c^T x & \text{D}: & \min y^T b \\ & Ax \leq \mathbf{b} & & A^T y = \mathbf{c} \\ & & & y \geq \mathbf{0} \end{array}$$

The duality theorem in its complete avatar states the following.

- If P is feasible and has a finite maximum then D is feasible and the two optimum values coincide.
- If P is infeasible and D is feasible then D is unbounded.
- If P feasible and unbounded then D is infeasible.

Exercise: Prove the theorem using the discussion in the last lecture and the duality theorem.

1 Duality from Lower Bounds

Consider the following LP.

$$\begin{aligned} \max\ & 14x_1 + 7x_2 + 22x_3 + 10x_4 \\ \text{s.t. } & 10x_1 + 3x_2 + 10x_3 + 7x_4 \leq 20 && (1) \\ & 3x_1 - 12x_2 - 13x_3 + 14x_4 \leq 35 && (2) \\ & 4x_1 + 4x_2 + 12x_3 + 3x_4 \leq 4 && (3) \end{aligned}$$

Looking at the above equations closely we can see that an upper bound on the objective is 24. This is because if we add (1) and (3), we get $14x_1 + 7x_2 + 22x_3 + 10x_4 \leq 24$. We could have multiplied the above equations by any non-negative factors (if we multiply by negative factors, the direction of the inequalities changes which we do not want) and then added to get an upper bound of the objective. This observation gives us another way of solving the LP.

Let us consider the following LP. Call it the primal (P).

$$\begin{aligned} \max\ & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t. } & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \quad \} \times y_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \quad \} \times y_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \quad \} \times y_m \end{aligned}$$

Suppose we are able to determine non-negative multipliers $y_1, y_2, \cdots, y_m$ such that, when the equations are multiplied by the respective multipliers and added together, we get the objective function on the left hand side. That is,

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &= c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &= c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &= c_n \\ y_i &\geq 0 \qquad\qquad \forall i = 1..m \end{aligned}$$

Then the sum of the right hand sides *i.e.* $b_1y_1 + b_2y_2 + \cdots + b_my_m$ gives an upper bound for the objective function of the primal. We would like the smallest upper bound. Consider all possible sets of multipliers satisfying the above requirement. Consider the upper bound given by each such set. The least among these upper bounds actually gives the optimum of the primal. (This is what the duality theorem says.) In other words the optimum objective of the primal is the same as the optimum objective of the following LP. Call it the dual (D).

$$\begin{aligned} \min\ & b_1y_1 + b_2y_2 + \cdots + b_my_m \\ \text{s.t.}\ & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m = c_1 \\ & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m = c_2 \\ & \vdots \\ & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m = c_n \\ & y_i \geq 0 \qquad\qquad \forall i = 1..m \end{aligned}$$

Suppose P feasible and unbounded. In this case, we will not be able to find a set of multipliers with the requirement stated in the above discussion. The reason is as follows. Suppose we are able to find some set of multipliers $y_1^*, y_2^*, \cdots, y_m^*$. Then the objective of P should be bounded from above by $b_1y_1^* + b_2y_2^* + \cdots + b_my_m^*$. But, this contradicts the fact that P is unbounded.

2 Dealing with Degeneracy

We describe now how degeneracy is dealt with. This section is optional and you may skip it.

Suppose a set $\mathbb{H}$ of $k > n$ hyperplanes intersect at a point x. As before, simplex at any stage keeps track of n of these: call this A'. Note that we cannot say that the other inequalities are strict as we could in the non-degenerate case.

As usual, it will determine a direction to move based on columns of $-A'^{-1}$ and execute a procedure to move to a neighbour as before. This will give rise to a new set of n hyperplanes which differs from A' in one row. This may yield a new point different from x or it may be the same point x where one hyperplane of $\mathbb{H}$ is replaced by another.

There is a phenomenon called *cycling* where one can cycle among subsets of $\mathbb{H}$ without exiting the point x. In other words, simplex could start at one subset A', go through some other subsets and then come back to the subset A'. This is a problem. One can avoid this problem by using certain rules to break ties. For instance which direction to move in and once the direction is determined, which inequality to include in place of the inequality that will be left out.

Assuming a rule to break ties, which does not let simplex cycle, it is clear that simplex will terminate. In other words it will reach a vertex x_0, given by some A' such that the cost is non-decreasing along the columns of $-A'^{-1}$. The proof of optimality of simplex, and the proof of the duality theorem now follows along the same lines as before.

One way to perturb the system is to replace b_i by $b_i + \epsilon^i$. For small enough ϵ this system is non-degenerate. Why? Also if ϵ is small enough, the optimum vertex output will be optimum in the original too. Why? What is interesting is that this behaviour of simplex can be simulated without changing b by a suitable rule to break ties above. Can you figure this rule out?