CS 435 : Linear Optimization — Fall 2008

Lecture 9: Convex Hull of Extreme Points

Lecturer: *Sundar Vishwanathan*
Computer Science & Engineering — Indian Institute of Technology, Bombay

In this lecture, we complete the proof of the theorem on extreme points mentioned in the previous lecture and begin the last part of understanding the object $\{x \ : \ Ax \leq \mathbf{b}\}$.

Proof: (Continuing Part 2.) Here we prove that every extreme point of $\{x : Ax \leq \mathbf{b}\}$ can be expressed as an intersection of n linearly independent hyperplanes. We show that if a point in the set cannot be expressed as an intersection of n linearly independent hyperplanes then we can express it as a convex combination of two other points in the set.

Let x_0 be such a point. We split $Ax_0 \leq \mathbf{b}$ into two parts

$$A^{'}x_0 \ = \ \mathbf{b}^{'} \tag{1}$$
$$A^{''}x_0 \ < \ \mathbf{b}^{''} \tag{2}$$

By the assumption on x_0, A' has rank strictly less than n. We wish to express x_0 as the convex combination of two other points y and z. Where do we find this y and z ? At this point, we recommend visualising examples in two and three dimensions. It will not take long to conjecture that y and z must also satisfy $A'x = b'$. For instance, if the object is a cube, and x_0 is a point on one face of the cube, it is easy to see that both y and z must also be on that face.

We need to use the fact that $A^{'}$ has rank less than n. One fallout of this is that the solutions to $A^{'}x = b'$ is a subspace of dimension at least one (at least a line) shifted by a vector and x_0 lies in this subspace. Again it is natural to consider y and z on a line passing through x_0 in the subspace, very close to, and on either side of x_0. This is what we will show below.

Since $A^{''}x_0$ is strictly less than $\mathbf{b}^{''}$, we can draw a small enough sphere around x_0 such that every point x within the sphere satifies $A^{''}x_0 < \mathbf{b}^{''}$. This means that there is an ϵ such that for which all unit vectors $\mathbf{v}$, $\epsilon\mathbf{v}$,

$$A^{''}(x_0 + \epsilon\mathbf{v}) < \mathbf{b}^{''} \tag{3}$$

Since $A^{'}$ does not have n linearly independent vectors, $A^{'}x = 0$ will have a non zero solution. Let x' be a non zero solution of $A^{'}x = 0$. Then $A^{'}(x_0 + \epsilon x') = A^{'}(x_0 - \epsilon x') = \mathbf{b}^{'}$ Also,

$$A^{''}(x_0 + \epsilon x') \ < \ \mathbf{b}^{''} \tag{4}$$
$$A^{''}(x_0 - \epsilon x') \ < \ \mathbf{b}^{''} \tag{5}$$

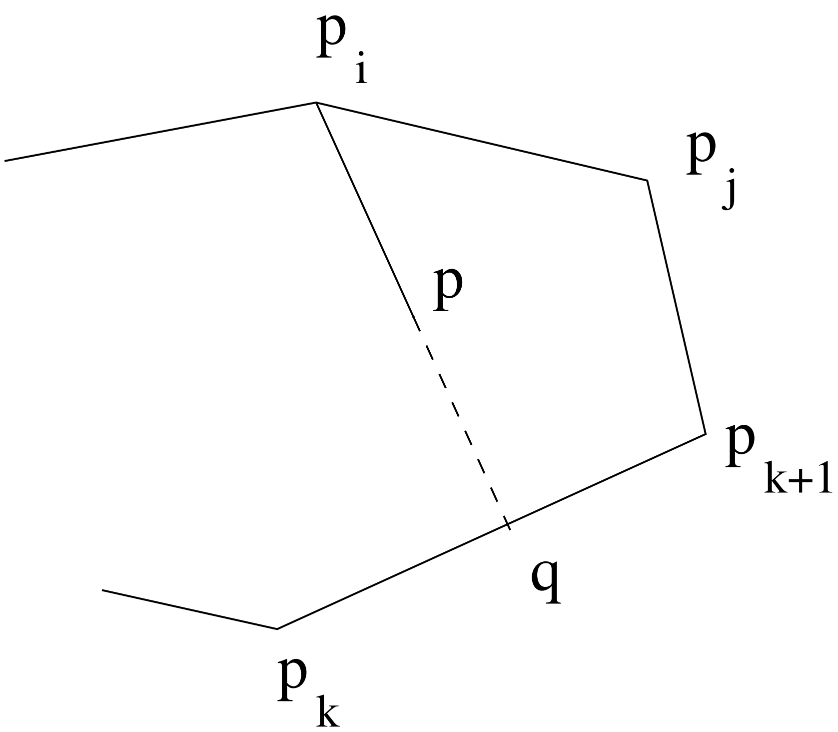
So $x_0 + \epsilon x'$ and $x_0 - \epsilon x'$ are points in $\{x : Ax \leq \mathbf{b}\}$ and $x_0 = (1/2)(x_0 + \epsilon x') + (1/2)(x_0 - \epsilon x')$. Thus we have expressed x_0 as a convex combination of two points in the set. $\square$

1 Convex hull of the extreme points

Definition 1 *A convex hull of points $p_1, p_2, \ldots p_n$ is the set of all points which can be written as convex combination of $p_1, p_2, \ldots p_n$.*

One can make a definition even when the set of points is infinite, but we will only deal with finite sets in this course.

The last step in understanding $\{x \ : \ Ax \leq \mathbf{b}\}$ is the following theorem.



THEOREM 1 *Let $p_1, p_2, \ldots p_n$ be extreme points of $x : Ax \leq \mathbf{b}$. Then every point in $x : Ax \leq \mathbf{b}$ can be expressed as a convex combination of the points $p_1, p_2, \ldots p_n$.*

Take any x_0 such that $Ax_0 \leq \mathbf{b}$. We need to show that this can be written as:

$$x_0 = \Sigma_i \lambda_i p_i \text{ where } \Sigma \lambda_i = 1 \ , \ 0 \leq \lambda_i \leq 1. \tag{6}$$

Again, it is instructive to look at examples especially in two dimensions and see what can be done.

Our proof will be by induction on the dimension of the object $\{x \ : \ Ax \leq \mathbf{b}\}$. *Base Case*: Exercise: Do this when the dimension is 0 and 1.

For the rest of this lecture, we will use an example in two dimensions to illustrate the technique of the proof. So consider a set $\{x : Ax \leq \mathbf{b}\}$ as shown in the figure.

Let p be any point inside the set. Now take the extreme point p_i, join it to p, and we extend this line till it touches one of the bounding segments ($p_k p_{k+1}$ in this case) at a point, say q.

Since q lies on the segment joining the extreme points p_k and p_{k+1} it can be expressed as a convex combination of p_k and p_{k+1}. Therefore,

$$q = \lambda_1 p_k + \lambda_2 p_{k+1} \text{ where } \lambda_1 + \lambda_2 = 1. \tag{7}$$

Also p lies on the segment joining q and p_i. So p can expressed as a convex combination of q and p_i. Therefore,

$$p = \lambda_3 p_i + \lambda_4 q \text{ where } \lambda_3 + \lambda_4 = 1. \tag{8}$$

Combining the above equations we get

$$p = \lambda_3 p_i + \lambda_4 \lambda_1 p_k + \lambda_4 \lambda_2 p_{k+1} \tag{9}$$

Now $\lambda_3 + \lambda_4 \lambda_1 + \lambda_4 \lambda_2 = 1$. Why?

Thus we have expressed p as a convex combination of the extreme points. Where did we use induction ? In expressing q as a convex combination of p_k and p_{k+1}. Once we have q, our focus was the line segment between p_k and p_{k+1}.

In the next lecture we use the same technique and complete the proof. This needs some clear thinking and the ability to be able to express your intuition in the language we have developed.