In this lecture, we complete the proof of the theorem on extreme points mentioned in the previous lecture and begin the last part of understanding the object \( \{ x : Ax \leq b \} \).

**Proof:** (Continuing Part 2.) Here we prove that every extreme point of \( \{ x : Ax \leq b \} \) can be expressed as an intersection of \( n \) linearly independent hyperplanes. We show that if a point in the set cannot be expressed as an intersection of \( n \) linearly independent hyperplanes then we can express it as a convex combination of two other points in the set.

Let \( x_0 \) be such a point. We split \( Ax_0 \leq b \) into two parts

\[
A'x_0 = b' \quad (1) \\
A''x_0 < b'' \quad (2)
\]

By the assumption on \( x_0 \), \( A' \) has rank strictly less than \( n \). We wish to express \( x_0 \) as the convex combination of two other points \( y \) and \( z \). Where do we find this \( y \) and \( z \)? At this point, we recommend visualising examples in two and three dimensions. It will not take long to conjecture that \( y \) and \( z \) must also satisfy \( A'x = b' \). For instance, if the object is a cube, and \( x_0 \) is a point on one face of the cube, it is easy to see that both \( y \) and \( z \) must also be on that face.

We need to use the fact that \( A' \) has rank less than \( n \). One fallout of this is that the solutions to \( A'x = b' \) is a subspace of dimension at least one (at least a line) shifted by a vector and \( x_0 \) lies in this subspace. Again it is natural to consider \( y \) and \( z \) on a line passing through \( x_0 \) in the subspace, very close to, and on either side of \( x_0 \). This is what we will show below.

Since \( A''x_0 \) is strictly less than \( b'' \), we can draw a small enough sphere around \( x_0 \) such that every point \( x \) within the sphere satisfies \( A''x_0 < b'' \). This means that there is an \( \epsilon \) such that for which all unit vectors \( v, \epsilon v \),

\[
A''(x_0 + \epsilon v) < b'' \quad (3)
\]

Since \( A' \) does not have \( n \) linearly independent vectors, \( A'x = 0 \) will have a non zero solution. Let \( x' \) be a non zero solution of \( A'x = 0 \). Then \( A'(x_0 + \epsilon x') = A'(x_0 - \epsilon x') = b' \). Also,

\[
A''(x_0 + \epsilon x') < b'' \quad (4) \\
A''(x_0 - \epsilon x') < b'' \quad (5)
\]

So \( x_0 + \epsilon x' \) and \( x_0 - \epsilon x' \) are points in \( \{ x : Ax \leq b \} \) and \( x_0 = (1/2)(x_0 + \epsilon x') + (1/2)(x_0 - \epsilon x') \). Thus we have expressed \( x_0 \) as a convex combination of two points in the set. \( \square \)

### 1 Convex hull of the extreme points

**Definition 1** A convex hull of points \( p_1, p_2, \ldots, p_n \) is the set of all points which can be written as convex combination of \( p_1, p_2, \ldots, p_n \).

One can make a definition even when the set of points is infinite, but we will only deal with finite sets in this course.

The last step in understanding \( \{ x : Ax \leq b \} \) is the following theorem.
Theorem 1 Let $p_1, p_2, \ldots, p_n$ be extreme points of $x: Ax \leq b$. Then every point in $x: Ax \leq b$ can be expressed as a convex combination of the points $p_1, p_2, \ldots, p_n$.

Take any $x_0$ such that $Ax_0 \leq b$. We need to show that this can be written as:

$$x_0 = \sum \lambda_i p_i \quad \text{where} \quad \Sigma \lambda_i = 1, \ 0 \leq \lambda_i \leq 1. \quad (6)$$

Again, it is instructive to look at examples especially in two dimensions and see what can be done.

Our proof will be by induction on the dimension of the object $\{x : Ax \leq b\}$. Base Case: Exercise: Do this when the dimension is 0 and 1.

For the rest of this lecture, we will use an example in two dimensions to illustrate the technique of the proof. So consider a set $\{x : Ax \leq b\}$ as shown in the figure.

Let $p$ be any point inside the set. Now take the extreme point $p_i$, join it to $p$, and we extend this line till it touches one of the bounding segments $(p_k, p_{k+1}$ in this case) at a point, say $q$.

Since $q$ lies on the segment joining the extreme points $p_k$ and $p_{k+1}$ it can be expressed as a convex combination of $p_k$ and $p_{k+1}$. Therefore,

$$q = \lambda_1 p_k + \lambda_2 p_{k+1} \quad \text{where} \quad \lambda_1 + \lambda_2 = 1. \quad (7)$$

Also $p$ lies on the segment joining $q$ and $p_i$. So $p$ can expressed as a convex combination of $q$ and $p_i$. Therefore,

$$p = \lambda_3 p_i + \lambda_4 q \quad \text{where} \quad \lambda_3 + \lambda_4 = 1. \quad (8)$$

Combining the above equations we get

$$p = \lambda_3 p_i + \lambda_4 \lambda_1 p_k + \lambda_4 \lambda_2 p_{k+1} \quad (9)$$

Now $\lambda_3 + \lambda_4 \lambda_1 + \lambda_4 \lambda_2 = 1$. Why?

Thus we have expressed $p$ as a convex combination of the extreme points. Where did we use induction? In expressing $q$ as a convex combination of $p_k$ and $p_{k+1}$. Once we have $q$, our focus was the line segment between $p_k$ and $p_{k+1}$.\[\]}
In the next lecture we use the same technique and complete the proof. This needs some clear thinking and the ability to be able to express your intuition in the language we have developed.