CS206 Homework #2

Max marks: 70 **Due Mar 20, 2007**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Do not copy solutions from others.***

1. *[20 (+ bonus 10) marks]* In a sleepy little town called Cigol, there is an old postman who *must* visit each morning all houses/offices that have letter(s) to be delivered that day. There are N houses/offices in Cigol, and each house/office has a distinct address that is a natural number in the range $[1, N]$. The post office itself has the address P.

   The postman, being old, must figure out the "best" sequence of addresses for delivering letters each morning, so that the *total distance* travelled by him *from the post-office to the last address where a letter is delivered and back to the post-office* is minimized. It is assumed that if the postman is currently at address x_i and the next address in the sequence is x_j, then he always takes the shortest path from x_i to x_j. Note that the "best" sequence for delivering letters need not be unique, and there may be letter(s) addressed to the post-office itself.

   Interestingly, this is no ordinary post-office! It has recently acquired a software that can analyze certain classes of predicate logic sentences (formulae without free variables) and can compute models of satisfiable sentences using natural numbers (i.e., the domain of elements is the set of natural numbers). Our old postman, having realized the fun of playing with predicate logic, now wishes to use this software for finding the best sequence of delivery addresses each morning. We must help him in this noble endeavour. In particular, we must formulate a satisfiable predicate-logic sentence ϕ (containing no free variables), such that the "best" sequence of addresses can be "extracted" from a model of ϕ using natural numbers. In constructing ϕ, we are allowed to use the following predefined predicates and functions:

   - $\mathsf{L}(x_i)$: This predicate takes a natural number x_i as argument, and if x_i lies in the range $[1, N]$, then it evaluates to True iff there is a letter for address x_i. If x_i lies outside the range $[1, N]$, the predicate evaluates to False. Clearly, the definition of $\mathsf{L}(x_i)$ potentially changes every morning (depending on which addresses have letters to be delivered that day). Our postman, in his enthusiasm to use the software, has agreed to look at the pile of letters each morning and update the definition of this predicate on the post-office computer before running the software on ϕ.
   - $\mathsf{D}(x_i, x_j)$: This function takes two natural numbers and returns a natural number giving the shortest distance between addresses x_i and x_j in Cigol, for all $x_i, x_j \in \{1 \ldots N\}$. If either argument is outside the range $[1, N]$, the function returns (rather arbitrarily chosen) 231. Cigol is fortunately seismologically stable, and there are no new constructions or demolitions (unlike our own campus!). So the definition of this function is assumed to be time invariant, and does not change from one day to another

- $+(x_i, x_j)$: This function takes two natural numbers and returns their sum.
- $*(x_i, x_j)$: This function takes two natural numbers and returns their product.
- $x_i = x_j$: This is the equality predicate for natural numbers.
- $x_i \geq x_j$: This is the greater-than-or-equal-to predicate for natural numbers.

In addition to the above predefined predicates and functions, you are allowed to use the function symbol Seq of arity 1 (from natural numbers to natural numbers) in formulating ϕ. No other function or predicate symbols may be used in ϕ.

Recall that our noble goal was to to help the old postman "extract" the "best" sequence of addresses every morning. So, we must use the function symbol Seq in ϕ in such a way that if the post-office's software returns a model $M = (\mathbf{N}, \mathsf{Seq}^M)$ for ϕ (where $\mathbf{N}$ is the set of natural numbers and $\mathsf{Seq}^M : \mathbf{N} \to \mathbf{N}$ is a concrete function definition), then Seq^M can be used to obtain a "best" sequence of addresses for delivering letters. In particular, if there are K addresses that have letters to be delivered on a particular day, then $\mathsf{Seq}^M(j)$ must give the j^{th} address in a "best" sequence for that day, for all $j \in [1, K]$. For $j \notin [1, K]$, $\mathsf{Seq}^M(j)$ must be 0. You may assume that there is at least one letter to be delivered each day, i.e. $1 \leq K \leq N$ for each day, although K may vary from day to day.

Note that ϕ must not involve K, since this may indeed differ from day to day, and our old postman cannot enter a new formula to the software each day. N is assumed to be fixed, so you may use N and functions of N in formulating ϕ, but once formulated, our postman must be able to use the same formula ϕ every day, simply by changing the definition of $\mathsf{L}(x_i)$ each morning as described above.

As an example, suppose there are 20 houses/offices in Cigol, and on a particular day, there are letters for addresses $1, 2$ and 12. Suppose also that the unique best sequence of addresses (for minimizing the total distance travelled) is $(12, 2, 1)$. To find this sequence, the postman goes and updates the definiton of L so that $\mathsf{L}(1) = \mathsf{L}(2) = \mathsf{L}(12) = \mathsf{True}$ and $\mathsf{L}(x_i) = \mathsf{False}$ for all other x_i. Our formula ϕ should be such that with the above definiton of L, a model $M = (\mathbf{N}, \mathsf{Seq}^M)$ of ϕ has $\mathsf{Seq}^M(1) = 12, \mathsf{Seq}^M(2) = 2, \mathsf{Seq}^M(3) = 1$ and $\mathsf{Seq}^M(i) = 0$ for all other i.

Give a predicate logic sentence ϕ for the above purpose, and explain briefly the justification behind the construction of your formula. In your justification, you must argue why any model $(\mathbf{N}, \mathsf{Seq}^M)$ of ϕ must contain a function Seq^M that gives a "best" sequence of addresses.

Important: Your justification **must not** exceed 500 words, and your formula **must** have size at most quadratic in N. In other words, the total no. of symbols (variables, predicate-logic operators, function and predicate symbols, parentheses) in your formula must be $O(N^2)$.

To get the bonus points, your formula must have size linear in N, i.e. $O(N)$.

2. *[10 + 10 marks]* Let f be a unary function and $\mathcal{M}$ be a model containing an interpretation of f. In general, some elements x in the universe (or domain) of $\mathcal{M}$ satisfy $f(x) = x$, while other elements satisfy $\neg(f(x) = x)$.

   (a) We wish to obtain a predicate logic sentence ψ that evaluates to True in $\mathcal{M}$ if and only if there are infinitely many elements x in the universe of $\mathcal{M}$ that satisfy $f(x) = x$. Is it possible to have such a predicate logic sentence ψ? If your answer is in the affirmative, you must provide ψ. If your answer is in the negative, you must prove that such a sentence ψ cannot exist.

   (b) A logician wish to write a predicate logic sentence ϕ that evaluates to True in $\mathcal{M}$ if and only if the universe of $\mathcal{M}$ has more elements x satisfying $\neg(f(x) = x)$ than elements y satisfying $f(y) = y$. Show that such a predicate logic sentence does not exist.

Note that if sets A and B are both infinite, saying that A has more elements than B is equivalent to saying that there exists an injective function from B to A, but there exists no injective function from A to B.

3. *[10 + 10 + 10 marks]* Consider the predicate logic sentence $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$, where

$$\begin{aligned}
\phi_1 &= \forall x\, (f(g(x)) = g(x)),\\
\phi_2 &= \forall x\, (g(f(x)) = f(x)),\\
\phi_3 &= \forall x \exists y\, ((x = g(y)) \vee (x = f(y)))
\end{aligned}$$

(a) Let $\mathcal{H}$ be the Herbrand Universe for the sentence ϕ. Let $M_{\mathcal{H}}$ be a model with the universe (or domain) being $\mathcal{H}$, and with the obvious interpretations of functions f and g, i.e., if t is a term in $\mathcal{H}$, the terms $f(t)$ and $g(t)$ in $\mathcal{H}$, are defined to be the result of applying f and g respectively to t. If $M_{\mathcal{H}} \models \phi$, how many *distinct elements* can be present in the universe of $M_{\mathcal{H}}$. Give justification for your answer.

(b) Show using natural deduction for predicate logic that $\phi_1, \phi_2, \phi_3 \vdash (x = f(x))$.

(c) Let $\phi_4 = \exists x\, \neg(f(x) = g(x))$. Show using Herbrand's Theorem that $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$ is unsatisfiable. Your solution for this part must not use results obtained in previous parts.