CS208 HW #3

Due Date: Mar 8, 2012

- ***Be brief, complete and stick to what has been asked. If needed, you may cite results/proofs covered in class without reproducing them.***
- ***Do not copy solutions from others***
- ***Penalty for copying: FR grade***

1. *[10+10 marks]* A set of integers is said to be *linear* if it is of the form $\{p+q\cdot i \mid i \in \mathbf{N}\}$, for fixed $p, q \in \mathbf{N}$. For example, $\{3, 7, 11, 15, 19, \ldots\}$ is a linear set with $p = 3, q = 4$. A set of integers is said to be *semi-linear* if it is the *finite union* of linear sets. For example, $\{3, 4, 6, 7, 8, 10, 11, 12, 15, \ldots\}$ can be written as the union of $\{3, 7, 11, 15, \ldots\}$ and $\{4, 6, 8, 10, 12, \ldots\}$. Since both these sets are linear, the set $\{3, 4, 6, 7, 8, 10, 11, 12, 15, \ldots\}$ is semi-linear.

   Now consider an alphabet Σ and a language $L \subseteq \Sigma^*$. Define the spectrum of L as $Spec(L) = \{n \mid n \in \mathbf{N}, L \text{ has at least one string of length } n\}$.

   (a) Show that if L is regular, then $Spec(L)$ must be semi-linear.

   (b) Use the above result to show that $\{0^i 1^j \mid i + j = k^3 \text{ for some } k \in \mathbf{N}\}$ is not regular.

2. *[10 marks]* Let $D = \{0, 1\}$ and $\Sigma = D \times D \times D$. Thus, some representative elements of Σ are $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, etc. A correct multiplication of two natural numbers can be represented by a string in Σ^*. To see an illustration of this, consider the multiplication $10 \times 5 = 50$.

   This can be expressed as: $\begin{array}{rcccccc} & 0 & 0 & 1 & 0 & 1 & 0 \\ \times & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline & 1 & 1 & 0 & 0 & 1 & 0 \end{array}$

The above multiplication can now be represented by the following string of symbols from Σ: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

More formally, since every letter $a \in \Sigma$ has three components, let us use $a[0], a[1]$ and $a[2]$ to refer to the three components of a. A string $w = w_0 w_1 \ldots w_k \in \Sigma^*$ represents a correct multiplication of two natural numbers if the natural number represented by $w_0[2] w_1[2] \ldots w_k[2]$ is the result of multiplying the natural number represented by $w_0[0] w_1[0] \ldots w_k[0]$ with that represented by $w_0[1] w_1[1] \ldots w_k[1]$.

Let L be the set of strings in Σ^* that represent correct multiplication of two natural numbers. Show that L is not regular.

3. *[0 + 10 + 10 marks]* The solution to this problem is specific to your roll number. Let $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We will say a string $w \in \Sigma^*$ is *embedded* in a string $u \in \Sigma^*$ iff w can be obtained from u by replacing some letters in u with ε. Thus, 014 is embedded in 203421049. To see why this is so, notice that 014 can be obtained as $\varepsilon 0 \varepsilon\varepsilon\varepsilon 1 \varepsilon 4 \varepsilon$. However, 014 is not embedded in 23421049.

   (a) State your roll no. as a string over Σ. Let us call this string ρ.

   (b) Give a DFA with no more than $|\rho| + 1$ states that accepts the language $L_\rho = \{w \mid w \in \Sigma^*, \rho \text{ is embedded in } w\}$. You must explain what each state in your DFA represents (use a couple of lines of explanation per state), as evidence that you understood the construction of the DFA. Your answer will fetch 0 marks without proper explanation per state.

   Note: You can of course start with a NFA, and then use the subset construction to convert it to a DFA and then use the minimization construction to minimize the DFA. But this is a long and tedious route, and will not give you much intuition to understand what each state in the resulting DFA represents. So you are strongly advised not to follow this route. There is enough structure in the problem to permit a much simpler construction of a DFA, and you are encouraged to think about it.

   (c) Apply the DFA minimization algorithm studied in class to the DFA obtained in the previous subquestion to obtain the minimum state DFA for L_ρ.

4. *[5+5+5 marks]* For each of the following languages, either give a DFA accepting the language (if you think that the language is regular), or give a proof of non-regularity (if you think that the language is not regular).

   (a) $\{w \mid w \in \{0,1\}^*, \#1(w) > \#0(w)$ or there are two consecutive 0s in $w\}$. Here, $\#1(w)$ and $\#0(w)$ denote the number of 1's and 0's respectively in w.

   (b) $\{w \cdot w \mid w \in \{0,1\}^*\}$

   (c) $\{w \cdot u \cdot w \mid w, u \in \{0,1\}^*\}$.