Propositional Logic: Syntax and Semantics

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Semantics of Propositional Logic

Consider a formula φ with n variables. Let 0 represent "false" and 1 represent "true"

- $[\![\varphi]\!] : \{0,1\}^n \rightarrow \{0,1\}$
- **Semantics is a function**
  - Often represented in tabular form: Truth Table
- Indicates truth value of formula, given truth values of all variables

Rules of semantics

- $[\![\neg\varphi]\!] = 1$ iff $[\![\varphi]\!] = 0$.
- $[\![\varphi_1 \wedge \varphi_2]\!] = 1$ iff $[\![\varphi_1]\!] = [\![\varphi_2]\!] = 1$.
- $[\![\varphi_1 \vee \varphi_2]\!] = 1$ iff at least one of $[\![\varphi_1]\!]$ or $[\![\varphi_2]\!]$ evaluates to 1.
- $[\![\varphi_1 \rightarrow \varphi_2]\!] = 1$ iff at least one of $[\![\varphi_1]\!] = 0$ or $[\![\varphi_2]\!] = 1$.
- $[\![\varphi_1 \leftrightarrow \varphi_2]\!] = 1$ iff both $[\![\varphi_1 \rightarrow \varphi_2]\!] = 1$ and $[\![\varphi_2 \rightarrow \varphi_1]\!] = 1$.

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  - φ is valid iff $\neg\varphi$ is unsatisfiable.

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  - Semantic equivalence implies equisatisfiability, not vice versa

Fun with semantics!

Two syntactically different formulas may be semantically equivalent!

E.g. $\varphi_1 : p \rightarrow (q \rightarrow r)$, $\varphi_2 : (p \wedge q) \rightarrow r$, and $\varphi_3 : (q \wedge \neg r) \rightarrow \neg p$

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Works, but doesn't scale! 2^n rows for n propositions

Semantic reasoning without truth tables?

Yes, **proof rules**