CS 208 : Automata Theory and Logic	DATE 18/04/24

Lecture - 40

Topic: Semantic Relations in FOL

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1 Semantics of FOL

In the previous class we have seen and formalized how to check $M, \alpha \models \phi$(i.e Given structure M and binding α, does ϕ evaluate to true).Now we will see an example on it.

1.1 Semantics of FOL: Illustration

Lets take the following example:

$$\phi = \exists x R(x, f(y, a)) \rightarrow \exists z(\neg(z = a) \wedge R(z, y))$$

SO, we have to decide our M and α first:

1. The given Formula contains the following variables x,y,z.
2. We can clearly see $\exists x$ and $\exists z$ but nothing for y.SO,free(ϕ)(i.e free variables of ϕ) contains only y.
3. Now lets decide $\alpha(y)$ as 2.
4. The vocabulary for the given formula is {a, f , =, R}.
5. Now for every item in vocabulary i have to choose a mapping in M.Lets consider the following Mapping

   $$M = (N, (0, u + v, u < v))$$

   where:

   $$U^M \rightarrow N$$

   $$a^M \rightarrow 0$$

   $$f^M(u, v) \rightarrow u + v$$

   $$R^M(u, v) \rightarrow u < v$$

   $$= \rightarrow u = v$$

6. We know that if M,$\alpha \models \exists x\phi$ it can be written as M,$\alpha[x \rightarrow c] \models \phi$ based on this first lets evaluvate the term

   $$M, \alpha[z \rightarrow 1] \models (\neg(z = a) \wedge R(z, y))$$

   (i) We kept z as 1 and a is mapped 0 so z is not equal to a which means ¬(z = a) is true.
   (ii)We kept z as 1 and y is mapped 2(based on α) so z is less than y which implies R(z,y)(i.e

z<y) is true.
(iii)As both are true thier intersection is true.Which implies the given statement is true.
(iv)From the prinicples we already discussed in starting of sixth point the given formula can be re-written as

$$M, \alpha \models \exists z(\neg(z = a) \wedge R(z, y))$$

This means the RHS of the implies is true.

7. Now lets check the LHS part of the implies and see if it evaluvates to true.

8. The formula is

$$M, \alpha[x \to 0] \models R(x, f(y, a))$$

As we now y is mapped to 2 and a is mapped to 0 f(y,a) is y+a which is 2 and x(whose value is 0) is less than 2,so R also evaluates to true as x<f(y,a).The given one is true and can be rewritten as

$$M, \alpha \models \exists x R(x, f(y, a))$$

9. We finally got

$$M, \alpha \models \exists x R(x, f(y, a))$$

$$M, \alpha \models \exists z(\neg(z = a) \wedge R(z, y))$$

This clearly implies we proved $M, \alpha \models \phi$

10. You can do the same this taking $\alpha(y)$ as 1, then we get M,$\alpha \not\models \phi$ because if y mapped to 1,For R(z,y) to be true z must be 0, then if z is 0 it implies z is equal to a.So $\neg(z = a)$ evaluates to false .Then the RHS of implies in the given formula evaluates to false but LHS evaluates to true.

2 Semantic Relations in FOL

Let F = {ϕ1, ϕ2, . . .} be a (possibly infinite) set of formulas, and ψ be a formula

2.1 Semantic Entailment

F $\models \psi$ holds iff whenever M, $\alpha \models \phi_i$ for all $\phi_i \in$. F, then M, $\alpha \models \psi$ as well.
Example:

$$\{\forall x((x = a) \vee R(x, y)), R(a, y)\} \models \forall z R(z, y)$$

if any M,α is such that $M, \alpha \models \forall x((x = a) \vee R(x, y))$ this tells that $M, \alpha \models R(x, y)$ for all x except when x=a because x=a might be evaluated to true and R(x, y) might be evaluated to false.But using the second formula R(a,y) we can tell that $M, \alpha \models R(x, y)$ even when x=a.so we proved that $M, \alpha \models R(x, y)$ for all x(whether x is a or not a).so we proved

$$M, \alpha \models \forall z R(Z, y)$$

2.2 Satisfiability

ψ is satisfiable iff there is some M and α such that $M, \alpha \models \psi$
Example:

$$\psi = \exists x R(x, f(y, a)) \to \exists z(\neg(z = a) \wedge R(z, y))$$

This is the first example we took in section 1 and we also choosen M,α for which $M, \alpha \models \psi$ So the above formula is **satisfiable**.

2.3 Validity

A V-formula ψ is valid iff $M, \alpha \models \psi$ for all V-structures M and all bindings α that assign values from U^M to free(ψ).
Example:

$$\psi = \forall x P(x, y) \rightarrow \exists x P(x, y)$$

if M,$\alpha \models \forall x P(x, y)$ it can be written as M,$\alpha \models \neg(\exists x \neg(P(x, y)))$.Now going further it can be written as M,$\alpha[x \rightarrow c] \models \neg(\neg(P(x, y)))$.This can be converted into

$$M, \alpha \models \exists x P(x, y)$$

.We finally got if M,$\alpha \models \forall x P(x, y)$ then M,$\alpha \models \exists x P(x, y)$.Similarly we can prove if M,$\alpha \not\models \forall x P(x, y)$ then M,$\alpha \not\models \exists x P(x, y)$.Therefore for any M,$\alpha$ choosen M,$\alpha \models \psi$.SO the given formula is valid.

2.4 Consistency

F is consistent iff there is at least one M and α such that $M, \alpha \models \phi_i$ for all $\phi_i \in$ F
Example:

$$F = \{\exists x R(x, y), \exists x R(f(x), y), \exists x R(f(f(x)), y), ...\}$$

For the given F it is very clear that the free variable is y for all the ϕ_i.So lets take

$$\alpha(y) = 2$$

$$M = (N, (u, u < v))$$

where:

$$U^M \rightarrow N$$

$$f^M(u) \rightarrow u$$

$$R^M(u, v) \rightarrow u < v$$

These M,α binds to all ϕ_i taking x$\rightarrow$0.Therefore,the given F is **consistent**.

3 Semantic Equivalence in FOL

$$\phi \equiv \psi \text{ iff } \{\phi\} \models \psi \text{ and } \{\psi\} \models \phi$$

where
$\{\phi\} \models \psi$ means that the set $\{\phi\}$ **semantically entails** ψ(we defined semantically entails in 2.1)

3.1 Quantifier equivalences

Here are some of the quantifier equivalences:

- $\forall x \forall y \psi \equiv \forall y \forall x \psi$
- $\exists x \exists y \psi \equiv \exists y \exists x \psi$
- $\forall x(\psi_1 \wedge \psi_2) \equiv (\forall x \psi_1) \wedge (\forall x \psi_2)$
- $\exists x(\psi_1 \vee \psi_2) \equiv (\exists x \psi_1) \vee (\exists x \psi_2)$
- If x $\notin$ free(ϕ_2), then Qx (ϕ_1 op ϕ_2) $\equiv$ (Qx ϕ_1) op ϕ_2, where Q $\in \{\exists, \forall\}$ and op $\in \{\vee, \wedge\}$.

Example(for the last one):

$$\forall x(\psi_1(x) \vee \psi_2(x)) \not\equiv (\forall x \psi_1(x)) \vee (\forall x \psi_2(x))$$

where as

$$\forall x(\psi_1(x) \vee \psi_2(y)) \equiv (\forall x \psi_1(x)) \vee (\psi_2(y))$$

because x $\notin$ free(ϕ_2)
Renaming Quantified Variables: Let z $\notin$ free(ϕ) $\vee$ bnd(ϕ). Then Qx $\phi \equiv$ Qz ϕ[z/x] for Q $\in \{\exists, \forall\}$
Example 1:

$$\exists x R(f(x,y),w) \equiv \exists z R(f(z,y),w)$$

Example 2:

$$\forall x \psi_1(x,y) \vee \forall x \psi_2(z,x) \equiv \forall x \psi_1(x,y) \vee \forall u \psi_2(z,u)$$

that can be further reduced to:

$$\forall x \psi_1(x,y) \vee \forall u \psi_2(z,u) \equiv \forall x(\forall u(\psi_1(x,y) \vee \psi_2(z,u)))$$

3.2 Negation Normal Form

We can Push negations down to atomic predicates using Demorgans law and using

$$\neg \exists x \phi(x) \equiv \forall x \neg \phi(x)$$

$$\neg \forall x \phi(x) \equiv \exists x \neg \phi(x)$$

3.3 Pull quantifiers out to the left

We will pull the quantifiers to the left using renaming as shown in example 2 in 3.1 if we are not able to do it directly.Here are some of the examples:

- $\exists x \phi(x) \vee \exists x \psi(x) \equiv \exists x(\phi(x) \vee \psi(x))$
- $\exists x \phi(x) \wedge \exists z \psi(z) \equiv \exists x \exists z(\phi(x) \wedge \psi(z))$
- $\forall x \phi(x) \wedge \exists x \psi(x) \equiv \forall x(\phi(x) \wedge \psi(x))$
- $\forall x \phi(x) \vee \forall z \psi(z) \equiv \forall x \forall z(\phi(x) \vee \psi(z))$