CS208 Tutorial Solutions

2024

Contents

Tutorial 1

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1. Determine if the following formulae are tautology.

   a) $(p \to q) \wedge (q \to r) \to (p \to r)$

   b) $(p \to q) \wedge (q \to r) \to (r \to p)$

   c) $(p_1 \wedge p_2 \ldots p_n) \to (p_1 \vee p_2 \ldots p_n)$

Solution:

a) Yes. We can show this by using truth table as follows.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
true	true	true	true	true	true
true	true	false	true	false	true
true	false	true	false	true	true
true	false	false	false	true	true
false	true	true	true	true	true
false	true	false	true	false	true
false	false	true	true	true	true
false	false	false	true	true	true

b) No. Its not tautology as for p as false and q as true and r as true, we can show that $(p \to q) \wedge (q \to r) \to (r \to p)$ is false

c) Yes. It is a tautology. However, we cannot verify this by drawing a truth table as it will become excessively cumbersome. Let us assume it is not a tautology, which means there exists an assignment for which $(p_1 \wedge p_2 \ldots p_n)$ evaluates to $\top$ and $(p_1 \vee p_2 \ldots p_n)$ evaluates to $\bot$. Any assignment for which $(p_1 \wedge p_2 \ldots p_n)$ evaluates to $\top$ must assign p_1 as $\top$. If p_1 is $\top$, then $(p_1 \vee p_2 \ldots p_n)$ must also evaluate to $\top$. Hence, we have a contradiction. Hence, the given formula is a tautology

Key Takeaway: Truth Tables act as the first principle for proving certain properties of (a) given formula(e). However, using truth tables for verification becomes cumbersome, and hence, we desire a proof system. A proof system is a set of rules for constructing proofs, which will be covered in the next part of the course.

2. Consider a set S of size n. Recall that for a relation over S to be a partial order we require the following to hold:

   i Reflexive: $x \preceq x$ for all $x \in S$

   ii Transitive: $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in S$

   iii Antisymmetric: $x \preceq y$ and $y \preceq x$ implies $x = y$ for all $x, y \in S$.

Moreover, $x \in S$ is called a *maximal* element in $\preceq$ if $x \preceq y$ holds only for $y = x$.
Consider n^2 propositional variables $\{p_{ij}\}_{1 \leq i,j \leq n}$ for an enumeration of $S = (x_1, x_2, \ldots, x_n)$ where p_{ij} is set to 1 iff $x_i \preceq x_j$.
Give a formula over these variables which evaluates to $\top$ iff $\preceq$ is a partial order. Also give a formula which evaluates to $\top$ only when $\preceq$ has a maximal element.

Solution: For reflexivity the formula is

$$R = p_{11} \wedge p_{22} \ldots \wedge p_{nn}.$$

For transitivity, the formula is

$$T = \bigwedge_{1\le i,j,k\le n} p_{ij} \wedge p_{jk} \to p_{ik} = \bigwedge_{1\le i,j,k\le n} \neg p_{ij} \vee \neg p_{jk} \vee p_{ik}.$$

Observe that we don't really need to consider the cases when any 2 of i, j, k are the same since if the reflexivity clauses are satisfied:

$$\begin{aligned}
(i = j) &\quad p_{ii} \wedge p_{ik} \to p_{ik} \equiv \top \wedge p_{ik} \to p_{ik} \equiv \top \\
(j = k) &\quad p_{ij} \wedge p_{jj} \to p_{ij} \equiv p_{ij} \wedge \top \to p_{ij} \equiv \top \\
(i = k) &\quad p_{ij} \wedge p_{ji} \to p_{ii} \equiv p_{ij} \wedge p_{ji} \to \top \equiv \top
\end{aligned}$$

Antisymmetric nature is captured by

$$A = \bigwedge_{1\le i,j\le n, i\ne j} \neg(p_{ij} \wedge p_{ji}) = \bigwedge_{1\le i,j\le n, i\ne j} \neg p_{ij} \vee \neg p_{ji}.$$

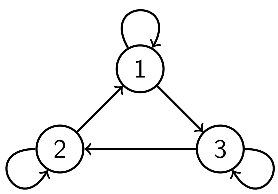
The final formula to check for a partial order then is: $R \wedge T \wedge A$.
If say x_i is a maximal element, we have $\bigwedge_{1\le j\le n, j\ne i} \neg p_{ij}$.
Hence, "has a maximal element" is encoded as

$$M = \bigvee_{1\le i\le n} \left(\bigwedge_{1\le j\le n, j\ne i} \neg p_{ij} \right).$$

Let us look at a concrete example with $n = 3$.

$$\begin{aligned}
R &= p_{11} \wedge p_{22} \wedge p_{33} \\
T &= (\neg p_{12} \vee \neg p_{23} \vee p_{13}) \\
&\wedge (\neg p_{13} \vee \neg p_{32} \vee p_{12}) \\
&\wedge (\neg p_{21} \vee \neg p_{13} \vee p_{23}) \\
&\wedge (\neg p_{23} \vee \neg p_{31} \vee p_{21}) \\
&\wedge (\neg p_{31} \vee \neg p_{12} \vee p_{32}) \\
&\wedge (\neg p_{32} \vee \neg p_{21} \vee p_{31}) \\
A &= (\neg p_{12} \vee \neg p_{21}) \\
&\wedge (\neg p_{23} \vee \neg p_{32}) \\
&\wedge (\neg p_{13} \vee \neg p_{31}) \\
M &= (\neg p_{12} \wedge \neg p_{13}) \\
&\vee (\neg p_{21} \wedge \neg p_{23}) \\
&\vee (\neg p_{31} \wedge \neg p_{32})
\end{aligned}$$

Consider the following relation which is not a partial order and doesn't have a maximal element.



Here $(p_{11} = 1, p_{12} = 0, p_{13} = 1, p_{21} = 1, p_{22} = 1, p_{23} = 0, p_{31} = 0, p_{32} = 1, p_{33} = 1)$. Substituting in values, you can confirm that R, A evaluate to $\top$ while T, M evaluate to $\bot$.

3. *Let n and k be integers so that $n > 0$, $k \geq 0$ and $k \leq n$. You are given n booleans x_1 through x_n and your goal is to come up with an efficient propositional encoding of the following constraint:*

$$\sum_{i=1}^{n} x_i \leq k \tag{1}$$

In the above equation, assume the booleans x_i behaves as 0 when false and 1 when true. We can arrive at the encoding by adding O(n·k) auxiliary variables and only need O(n·k) clauses. We try to solve this problem iteratively.

- *Let $s_{i,j}$ denote that at least j variables among $x_1, ..., x_i$ are assigned 1 (true). One can deduce that $s_{i,j}$ makes sense only when $i \geq j$. We now try to represent this constraint in propositional logic.*
- *Now given that you have represented $s_{i,j}$ for all appropriate $1 \leq i \leq n$ and $0 \leq j \leq k$; come up with a propositional formula that represents the condition of the equation (1)*

Solution:

- We start with simple observations; $x_i \leftrightarrow s_{i,1}$ $\forall$ i s.t. $1 \leq i \leq n$ and that $s_{1,j} = 0$ $\forall$ j s.t. $1 < j \leq k$
- We recursively can notice that, $s_{i-1,1} \rightarrow s_{i,1}$ $\forall$ i s.t. $1 < n$
- $x_i \wedge s_{i-1,j-1} \rightarrow s_{i,j}$ and $s_{i-1,j} \rightarrow s_{i,j}$ where i, j satisfy $1 < j \leq k$, $1 < i < n$
- Now to represent the final constraint we must make sure the following, $x_i \rightarrow \neg s_{i-1,k}$ for all i such that $1 < i \leq n$
- This could be viewed as a logic circuit similar to an adder circuit taught in your Digital Logic Design course

Note: For the constraint $\sum_{i=1}^{n} x_i \geq k$ there is an interesting encoding that uses $O(kn)$ auxiliary variables and $O(k^2 n)$ clauses using the Pigeon-Hole principle, as follows:
Let the x_i's define *holes* and let $x_i = 1$ mean that the ith hole is *fillable*. Our aim is to define p_{ij} for $i \in [k], j \in [n]$ such that it is 1 when *pigeon* i is assigned to hole j. The constraints we need to capture are:

$$\text{[fillable holes]} \quad \bigwedge_{i \in [k]} \bigwedge_{j \in [n]} (p_{ij} \rightarrow x_j)$$

$$\text{[only one pigeon per hole]} \quad \bigwedge_{\substack{i_1, i_2 \in [k] \\ i_1 \neq i_2}} \bigwedge_{j \in [n]} \neg(p_{i_1 j} \wedge p_{i_2 j})$$

$$\text{[each pigeon assigned to \textit{at least} 1 hole]} \quad \bigwedge_{i \in [k]} \bigvee_{j \in [n]} p_{ij}$$

Let f be the conjunction of these clauses, then:

$$\text{Satisfying assignment for } f \iff \text{at least } k \text{ holes fillable} \equiv \sum_{i=1}^{n} x_i \geq k$$

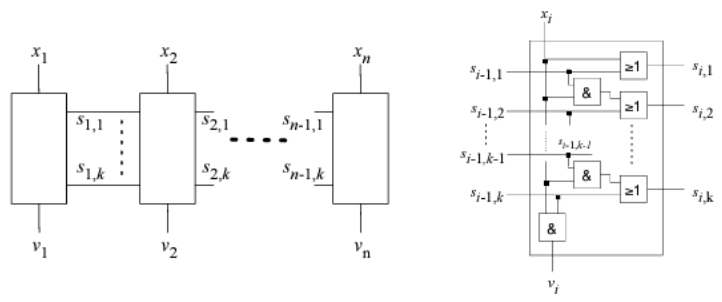


Figure 1: Logic Gate

4. Two friends find themselves trapped in a room. There are three doors coloured - red, blue and green respectively. Behind exactly one of the doors is a path to their home :-). The other two doors lead to horrible places. The inscriptions on the three doors are as follows:
Red door: "Your home town is not behind blue door"
Blue door: "Your home town is behind red door"
Green door: "Your home town is not behind blue door"
Given the fact that at LEAST ONE of the inscriptions is true and at LEAST ONE of them is false, which door would lead the boys home?

Solution: Let us use the following three propositional variables

- r: home town is behind red door
- b: home town is behind blue door
- g: home town is behind green door

Let us encode whatever we know

- behind one of the doors is a path to home, behind the other two doors is a way to ocean:
$$(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$$
- at least one of the three statements is true:
$$r \vee \neg b$$
- at least one of the three statements is false:
$$\neg r \vee b$$

r	b	g	2.5	2.6	$2.5 \wedge 2.6$
T	F	F	T	F	F
F	T	F	F	T	F
F	F	T	T	T	T

Thus, the friends should rush to the green door to get back to their home !!