Semantic Relations in FOL

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- **Semantic Entailment:** $\mathcal{F} \models \psi$ holds iff whenever $M, \alpha \models \varphi_i$ for all $\varphi_i \in \mathcal{F}$, then $M, \alpha \models \psi$ as well.
  - $\{\forall x\,((x = a) \vee R(x, y)), \quad R(a, y)\} \quad \models \quad \forall z\, R(z, y)$

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- **Satisfiability:** ψ is satisfiable iff there is some M and α such that $M, \alpha \models \psi$
  - $\exists x\, R(x, f(y, a)) \rightarrow \exists z\, (\neg(z = a) \wedge R(z, y))$ is satisfiable

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- **Validity:** A $\mathcal{V}$-formula ψ is valid iff $M, \alpha \models \psi$ for all $\mathcal{V}$-structures M and all bindings α that assign values from U^M to free(ψ).
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- **Consistency:** $\mathcal{F}$ is consistent iff there is at least one M and α such that $M, \alpha \models \varphi_i$ for all $\varphi_i \in \mathcal{F}$.
  - $\{\exists x\, R(x, y),\ \exists x\, R(f(x), y),\ \exists x\, R(f(f(x)), y), \ldots\}$ is consistent

Sematic Equivalence in FOL

$\varphi \equiv \psi$ iff $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Quantifier Equivalences

- $\forall x \forall y\, \varphi \;\equiv\; \forall y \forall x\, \varphi$, $\quad \exists x \exists y\, \varphi \;\equiv\; \exists y \exists x\, \varphi$
- $\forall x\, (\varphi_1 \wedge \varphi_2) \;\equiv\; (\forall x\, \varphi_1) \wedge (\forall x\, \varphi_2)$
- $\exists x\, (\varphi_1 \vee \varphi_2) \;\equiv\; (\exists x\, \varphi_1) \vee (\exists x\, \varphi_2)$
- If $x \not\in \mathsf{free}(\varphi_2)$, then $Qx\, (\varphi_1 \text{ op } \varphi_2) \;\equiv\; (Qx\, \varphi_1) \text{ op } \varphi_2$, where $Q \in \{\exists, \forall\}$ and op $\in \{\vee, \wedge\}$.

Renaming Quantified Variables

Let $z \not\in \mathsf{free}(\varphi) \cup \mathsf{bnd}(\varphi)$.
Then $Qx\, \varphi \;\equiv\; Qz\, \varphi[z/x]$ for $Q \in \{\exists, \forall\}$.

Enabler for substitution, e.g., $\exists x\, R(f(x, y), w) \;\equiv\; \exists z\, R(f(z, y), w)$
$f(x, y)$ not free for y in $\exists x\, R(f(x, y), w)$, but is free for y in $\exists z\, R(f(z, y), w)$.

Semantically Equivalent Transformations of FOL Formulae

Negation Normal Form

Push negations down to atomic predicates using

- DeMorgan's Laws
- $\neg\exists x\, \varphi(x) \equiv \forall x\, \neg\varphi(x)$ and $\neg\forall x\, \varphi(x) \equiv \exists x\, \neg\varphi(x)$ and

Pull quantifiers out to the left

- Rename every quantified variable to a fresh variable name
- Use rules for scoping of quantifiers in previous slide to pull all quantifiers out to the left
  - $\exists x\, \varphi(x) \;\vee\; \exists x\, \psi(x) \;\equiv\; \exists x\, (\varphi(x) \vee \psi(x))$
  - $\exists x\, \varphi(x) \;\wedge\; \exists z\, \psi(z) \;\equiv\; \exists x \exists z\, (\varphi(x) \wedge \psi(z))$
  - $\forall x\, \varphi(x) \;\wedge\; \forall x\, \psi(x) \;\equiv\; \forall x\, (\varphi(x) \wedge \psi(x))$
  - $\forall x\, \varphi(x) \;\vee\; \forall z\, \psi(z) \;\equiv\; \forall x \forall z\, (\varphi(x) \vee \psi(z))$

Prenex Normal Form (PNF)

First order logic formula of the form:

$$Q_1 x_1 Q_2 x_2 \ldots Q_k x_k \, \varphi(x_1, x_2, \ldots x_k, y_1, \ldots y_n)$$

$Q_i \in \{\exists, \forall\}$ for all $i \in \{1, \ldots k\}$ and $\varphi(\cdots)$ quantifier-free

- All quantifiers pulled out to the left: *quantifier prefix* of formula
  - Exact sequencing of $\forall$ and $\exists$ important
- $y_1, \ldots y_n$ are free variables
- $\varphi(x_1, x_2, \ldots x_k, y_1, \ldots y_n)$ is quantifier free: *matrix* of formula

Every FOL formula has a semantically equivalent PNF

Special prenex normal forms

- Prenex conjunctive normal form (PCNF): matrix in CNF w.r.t. atomic predicates
- Prenex disjunctive normal form (PDNF): matrix in DNF w.r.t. atomic predicates

Every FOL formula has a sem. equivalent PCNF and PDNF.

First-order Definable Structures

- If φ is a $\mathcal{V}$-sentence (no free vars), no binding α necessary for evaluating truth of φ
  - Given $\mathcal{V}$-structure M, we can ask if $M \models \varphi$
  - Class of $\mathcal{V}$-structures defined by φ is $\{M \models \varphi\}$
- Some examples of structures: graphs, databases, number systems

Graphs as FO structures

A graph G

- U^G: set of vertices
- Vocabulary $\mathcal{V}$: $\{E, =\}$, where E is a binary (edge) relation
- Interpretation: For $a, b \in U^G$, $E^G(a, b) =$ **true** iff there is an edge from vertex a to vertex b in G

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- $\forall x \forall y \forall z\, (\neg(x = y) \wedge \neg(y = z) \wedge \neg(z = x)) \rightarrow \neg(E(x, y) \wedge E(y, z) \wedge E(z, x))$
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  - (Infinite) class of all graphs with no cycles of length 3
- $\exists x \exists y\, (\neg(x = y) \wedge E(x, y) \wedge \forall z\, ((x = z) \vee (y = z)))$
  - (Finite) class of graphs with exactly two connected vertices.

Relational Databases as FO structures

A relational database D

- U^D: set of (possibly differently typed) data items
- Vocabulary $\mathcal{V}$: $\{P_1, \ldots P_k, =\}$, where P_i is a k_i-ary predicate corr. to the i^{th} table in database with k_i columns
- Interpretation: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) =$ **true** iff $(a_1, \ldots a_{k_1})$ is a row of the i^{th} table

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Example database query:

- $$\varphi(x) \triangleq \exists y \exists z\, (\mathsf{Dob}(x, y) \wedge \mathsf{After}(y, \text{“}01/01/1990\text{''}) \wedge \mathsf{Class}(x, z) \wedge \mathsf{Primary}(z))$$

  Defines set of students born after “01/01/1990” and studying in a primary class.

Number systems as FO structures

Natural/real numbers with addition, multiplication, linear ordering and constants $\mathbf{0}$ and $\mathbf{1}$ (fixed interpretation)

- $\mathfrak{N} = (\mathbb{N}, \mathbf{0}, \mathbf{1}, \times, +, <, =)$
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- $\mathfrak{R} \models \forall x \exists y\, (x = ((y \times y) \times y))$
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    - Not every natural number can be expressed as the sum of squares of two natural numbers. This can be done for real numbers
- $\mathfrak{N} \models \forall x \exists y\, ((x < y) \wedge (\forall z \forall w\, (y = z \times w) \rightarrow ((z = y) \vee (w = y)))$
    - There are infinitely many prime natural numbers