

First Order Logic: A Brief Introduction

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- Variables: x, y, z, \dots
 - Represent elements of an underlying set
- Constants: a, b, c, \dots
 - Specific elements of underlying set
- Function symbols: f, g, h, \dots
 - *Arity* of function: # of arguments
 - 0-ary functions: constants
- Relation (predicate) symbols: P, Q, R, \dots
 - Hence, also called “predicate calculus”
 - *Arity* of predicate: # of arguments
- Fixed symbols:
 - Carried over from prop. logic: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, (,)$
 - New in FOL: \exists, \forall (“quantifiers”)

Equality in FOL

- A special binary predicate, used widely in maths
- Represented by special predicate symbol “=”
- Semantically, binary identity relation (more on this later ...)
- First-order logic with equality
 - Different expressive power vis-a-vis first-order logic
 - Most of our discussions will assume availability of “=”
 - Refer to as “first-order logic” unless the distinction is important

Two classes of syntactic objects: *terms* and *formulas*

Terms

- Every variable is a term
- If f is an m -ary function, t_1, \dots, t_m are terms, then $f(t_1, \dots, t_m)$ is also a term

Atomic formulas

- If R is an n -ary predicate, t_1, \dots, t_n are terms, then $R(t_1, \dots, t_n)$ is an atomic formula
- Special case: $t_1 = t_2$

- *Primitive* fixed symbols: \wedge, \neg, \exists
 - Other choices also possible: E.g., $\vee, \rightarrow, \forall$

Rules for formulating formulas

- Every atomic formula is a formula
- If φ is a formula, so are $\neg\varphi$ and (φ)
- If φ_1 and φ_2 are formulas, so is $\varphi_1 \wedge \varphi_2$
- If φ is a formula, so is $\exists x \varphi$ for any variable x
- Formulas with other fixed symbols definable in terms of formulas with primitive symbols.
 - $\varphi_1 \vee \varphi_2 \triangleq \neg(\neg\varphi_1 \wedge \neg\varphi_2)$
 - $\varphi_1 \rightarrow \varphi_2 \triangleq \neg\varphi_1 \vee \varphi_2$
 - $\varphi_1 \leftrightarrow \varphi_2 \triangleq (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$
 - $\forall x \varphi \triangleq \neg(\exists x \neg\varphi)$

FOL formulas as strings

- Alphabet (over which strings are constructed):
 - Set of variable names, e.g. $\{x_1, x_2, y_1, y_2\}$
 - Set of constants, functions, predicates, e.g. $\{a, b, f, =, P\}$
 - Fixed symbols $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall\}$
- Well-formed formula: string formed according to rules on prev. slide
 - $\forall x_1(\forall x_2(((x_1 = a) \vee (x_1 = b)) \wedge \neg(f(x_2) = f(x_1))))$ is well-formed
 - $\forall(\forall x_1(x_1 = ab)\neg())x_2$ is not well-formed
- Well-formed formulas can be represented using parse trees
 - Consider the rules on prev. slide as production rules in a context-free grammar

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 - $\{a, b, f, =\}$

Free Variables in a Formula

Free variables are those that are not quantified in a formula.
Let $\text{free}(\varphi)$ denote the set of free variables in φ

- If φ is an atomic formula, $\text{free}(\varphi) = \{x \mid x \text{ occurs in } \varphi\}$
- If $\varphi = \neg\psi$ or $\varphi = (\psi)$, $\text{free}(\varphi) = \text{free}(\psi)$
- If $\varphi = \varphi_1 \wedge \varphi_2$, $\text{free}(\varphi) = \text{free}(\varphi_1) \cup \text{free}(\varphi_2)$
- if $\varphi = \exists x \varphi_1$, $\text{free}(\varphi) = \text{free}(\varphi_1) \setminus \{x\}$

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If φ has free variables $\{x, y\}$, we write $\varphi(x, y)$

A formula with no free variables is a **sentence**, e.g. $\exists x \forall y f(x) = y$

Bound Variables in a Formula

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 - $= \text{bnd}(P(x, y)) \cup \{x\} \cup \text{bnd}(Q(x, y)) \cup \{y\}$
 - $= \emptyset \cup \{x\} \cup \emptyset \cup \{y\}$
 - $= \{x\} \cup \{y\} = \{x, y\}$!!!
- $\text{free}(\varphi)$ and $\text{bnd}(\varphi)$ are not complements!

Substitution in FOL

Suppose $x \in \text{free}(\varphi)$ and t is any term.

We wish to replace every free occurrence of x in φ with t , such that free variables in t stay free in the resulting formula.

Term t is free for x in φ if no free occurrence of x in φ is in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t .

- $\varphi \triangleq \exists y R(x, y) \vee \forall x R(z, x)$, and t is $f(z, x)$
- $f(z, x)$ is free for x in φ , but $f(y, x)$ is not

$\varphi[t/x]$: Formula obtained by replacing each free occurrence of x in φ by t , if t is free for x in φ

- For φ defined above,
 $\varphi[f(z, x)/x] \triangleq \exists y R(f(z, x), y) \vee \forall x R(z, x)$

Semantics of FOL: Some Intuition

$$\varphi \triangleq \forall x \forall y (P(x, y) \rightarrow \exists z (\neg(z = x) \wedge \neg(z = y) \wedge P(x, z) \wedge P(z, y)))$$

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English reading: For every x and y , if $P(x, y)$ holds, we can find z distinct from x and y such that both $P(x, z)$ and $P(z, y)$ hold.

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Case 1:

- Variables take values from **real numbers**
- $P(x, y)$ represents **$x < y$**
- English reading simply states “real numbers are dense”
- φ is **true**

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Case 2:

- Variables take values from **real numbers**
- $P(x, y)$ represents $x \leq y$
- English reading requires the following to be true
 - If $x = y$, there is a z such that $z \neq x$, $x \leq z$ and $z \leq x$
 - Thus, $z \neq x$ and $z = x$
- φ is **false**

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Case 3:

- Variables take values from **natural numbers**
- $P(x, y)$ represents $x < y$
- English reading states that “natural numbers are dense”
- φ is **false**

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Case 3:

- Variables take values from **natural numbers**
- $P(x, y)$ represents $x < y$
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- φ is **false**

Truth of φ depends on the underlying set from which variables take values, and on how constants, functions, predicates are interpreted

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Vocabulary \mathcal{V} : E.g. $\mathcal{V} : \{a, f, =, R\}$

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e.g. *Interp. for* $= : \{(c, c) \mid c \in \mathbb{N}\}$ – fixed interpretation
Interp. for $R : \{(c, d) \mid c, d \in U, c < d\}$

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1 and 2 define a \mathcal{V} -**structure** $M = (U^M, (a^M, f^M, R^M))$

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- 3 *Binding (aka environment)* $\alpha : \text{free}(\varphi) \rightarrow U$
e.g. $\alpha(y) = 2$

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Given structure M and binding α , does φ evaluate to **true**?

Notationally, does $\mathbf{M}, \alpha \models \varphi$?

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 - If t is a variable x , $\bar{\alpha}(t) = \alpha(x)$
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 - In prev. example, suppose $\alpha'(x) = 1, \alpha'(y) = 2$. Then $M, \alpha' \models R(x, f(y, a))$ as $(\bar{\alpha}'(x), \bar{\alpha}'(f(y, a))) = (1, 2) \in R^M$.

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- $M, \alpha \models \neg\varphi_i$ iff $M, \alpha \not\models \varphi_i$

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- $M, \alpha \models \neg\varphi_i$ iff $M, \alpha \not\models \varphi_i$
- $M, \alpha \models \varphi_1 \wedge \varphi_2$ iff $M, \alpha \models \varphi_1$ and $M, \alpha \models \varphi_2$

Semantics of FOL: Formalizing the intuition

Given structure M and binding α , does φ evaluate to **true**?

Notationally, does $\mathbf{M}, \alpha \models \varphi$?

- Extend $\alpha : \text{free}(\varphi) \rightarrow U^M$ to $\bar{\alpha} : \text{Terms}(\varphi) \rightarrow U^M$
 - If t is a variable x , $\bar{\alpha}(t) = \alpha(x)$
 - If t is $f(t_1, \dots, t_m)$, $\bar{\alpha}(t) = f^M(\bar{\alpha}(t_1), \dots, \bar{\alpha}(t_m))$
 - In prev. example, $\bar{\alpha}(f(y, a)) = f^M(\alpha(y), a^M) = 2 + 0 = 2$
- If φ is an atomic formula
 - $M, \alpha \models (t_1 = t_2)$ iff $\bar{\alpha}(t_1)$ and $\bar{\alpha}(t_2)$ coincide
 - $M, \alpha \models P(t_1, \dots, t_m)$ iff $(\bar{\alpha}(t_1), \dots, \bar{\alpha}(t_m)) \in P^M$
 - In prev. example, suppose $\alpha'(x) = 1, \alpha'(y) = 2$. Then $M, \alpha' \models R(x, f(y, a))$ as $(\bar{\alpha}'(x), \bar{\alpha}'(f(y, a))) = (1, 2) \in R^M$.
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- $M, \alpha \models \exists x \varphi$ iff there is some $c \in U^M$ such that $M, \alpha[x \mapsto c] \models \varphi$, where
 - $\alpha[x \mapsto c](v) = \alpha(v)$, if variable v is different from x
 - $\alpha[x \mapsto c](x) = c$

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- Note that if $\alpha'(y) = 1$, $M, \alpha' \not\models \varphi$

Semantic Relations in FOL

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- **Semantic Entailment:** $\mathcal{F} \models \psi$ holds iff whenever $M, \alpha \models \varphi_i$ for all $\varphi_i \in \mathcal{F}$, then $M, \alpha \models \psi$ as well.
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- **Consistency:** \mathcal{F} is consistent iff there is at least one M and α such that $M, \alpha \models \varphi_i$ for all $\varphi_i \in \mathcal{F}$.
 - $\{\exists x R(x, y), \exists x R(f(x), y), \exists x R(f(f(x)), y), \dots\}$ is consistent