CS 228 : Logic in Computer Science

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- SAT solvers are tools which implement heuristics to check satisfiability.
- The input to a SAT solver is a formula in some specified form.

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Every formula F is equivalent to some formula F_1 in CNF and some formula F_2 in DNF.

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- Replace all subformulae of the form $F \to G$ with $\neg F \vee G$, and all subformulae of the form $F \leftrightarrow G$ with $(\neg F \vee G) \wedge (\neg G \vee F)$. When there are no more occurrences of $\to, \leftrightarrow$, proceed to the next step.

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- Get rid of all double negations, and push all negations inside to the level of literals : Replace all subformulae
  - $\neg\neg G$ with G,
  - $\neg(G \wedge H)$ with $\neg G \vee \neg H$
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  When there are no more such subformulae, proceed to the next step.
- Distribute $\vee$ wherever possible : that is, replace all $(F \vee (G \wedge H))$ or $((G \wedge H) \vee F)$ with $(F \vee G) \wedge (F \vee H)$.

The resultant formula F_1 is in CNF and is provably equivalent to F.
$[(\neg x \vee \neg y) \wedge (\neg x \vee \neg z)] \wedge [(\neg x \vee y) \wedge (\neg x \vee \neg x)]$

Polynomial Time Formula Classes

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- $p \wedge (\neg p \vee \neg q \vee r) \wedge (\neg a \vee \neg b)$ is Horn, but $a \vee b$ is not Horn.
- A basic Horn formula is one which has no $\wedge$. Every Horn formula is a conjunction of basic Horn formulae.

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- Thus, a Horn formula is written as a conjunction of implications.

A Decision Problem

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The class **P**

An algorithm is polynomial time if there exists a polynomial $p(x)$ such that given the input size n, the algorithm terminates with the correct answer in $\leqslant p(n)$ steps. The class of all problems which can be solved by a polynomial time algorithm is denoted **P**.

Horn SAT is in **P**

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- Consider subformulae of the form $(p_1 \wedge \cdots \wedge p_m) \to \bot$. If there is one such subformula with all p_i marked, then say Unsat, otherwise say Sat.

An Example

$$(\top \to A) \wedge (C \to D) \wedge ((A \wedge B) \to C) \wedge ((C \wedge D) \to \bot) \wedge (\top \to B).$$

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The Horn Algorithm

The Horn algorithm concludes Sat iff H is satisfiable.

Proof on the board

Complexity of Horn

- Given a Horn formula of length n, Horn SAT takes at most n^2 steps to conclude.
- Read once the formula symbol-by-symbol left to right and list all the atomic formulas (atmost n such)
- Read once marking all $\top \to p$ clauses.
- Inspect and mark all clauses $(p_1 \wedge \ldots p_j) \to Q$ (at most n times)
- Check one final time checking for a $\bot$ RHS and fully marked LHS

2-CNF

- 2-CNF : CNF where each clause has at most 2 literals.

$$(\neg p \vee q) \wedge p \wedge (r \vee \neg q) \wedge (\neg r \vee p)$$

2-CNF-SAT is also polytime. Think on the ideas we discussed in class towards this.