CS 228M : Logic for Computer Science Spring 2026

Practice Problem Set 4

Instructions:

- The following problems are meant for you to practice, so that your understanding of the topic improves.
- You must solve all problems to get the maximum benefit from practice problems.
- You must not submit your solutions to these problems. These are not going to be graded.
- A problem may have multiple solution techniques. Discussion among students is strongly encouraged in order to understand different perspectives.
- Questions marked * are comparably more difficult than the rest.

---

1. **Well-formed or not?**

   Using the rules for well formed formulas taught in class, determine if the formulas given below are well formed formulas or not.

   1. $\forall x(R(x,x) \rightarrow \exists y(P(x,y) \wedge P(f(x),y))$, where P is a binary relation, R is a binary relation, f is a unary function, and x, y are variables.
   2. $\forall x(R(x,x) \rightarrow \exists f(y)(P(x,y) \wedge P(f(x)),y))$, where the vocabulary is the same as in the above sub-problem.

   Carefully note the parenthesization in order to answer this question.

   (*Hint:* Recursive framework for well formed formulae is that, $\phi = \phi_1 \wedge \phi_2$ is well formed iff ϕ_1 and ϕ_2 are well formed, similarly for other operations too. Recursion ends when you reach atomic formulae (predicates))

2. **Additive Identity**

   Let U be a non-empty set over which binary operations $+$ and $\times$ are defined and closed (i.e. whenever these operations applied to elements of U, the result lies inside U). The structure $\mathcal{A} = (U, +, \times)$ is defined with some fixed interpretations of $+$ and $\times$ over U.

   An additive identity of U is defined as an element $t \in U$ such that $t + u = u + t = u$ for every element $u \in U$.

   For each sub-question below, write a first order logic sentence over the vocabulary $(=, +. \times)$ such that your sentence evaluates to true on the $\mathcal{A}$ iff the structure has the following properties:

   1. There exists an additive identity for U
   2. If the additive identity if exists, it is unique for U
   3. If the additive identity exists and is unique for U, then $t \times u = t$ holds for all $u \in U$ where t is the additive identity. (Note that you do not have a constant in the vocabulary to represent the additive identity; so you must encode it in the formula itself)

   For notational convenience, you can write $+(t_1, t_2)$ as $t_1 + t_2$, $\times(t_1, t_2)$ as $t_1 \times t_2$, and $= (t_1, t_2)$ as $t_1 = t_2$ for purposes of this question, where t_1 and t_2 are terms.

3. **FOL Parsing**
   Let the vocabulary $V = \{P, R, f, c\}$, where:
   - P and R are binary predicate symbols.
   - f is a unary function symbol.
   - c is a constant symbol.

   Consider the following First-Order Logic formula, φ:

$$\varphi := \forall x\Big(P(x, y) \to \exists y\big(R(x, f(y)) \land P(y, z)\big)\Big) \lor R(f(y), z)$$

   1. **Parse Tree:** Draw the full parse tree for φ.
   2. **Free variable Analysis:** Using the parse tree generated in the previous part, identify if the given formula is a sentence. If not, identify the free variables.
   3. **Substitution:** Determine if the following terms can be substituted for the free occurrences of y. If a term is *not* substitutable, give reasons:
      (a) $t_2 = f(z)$
      (b) $t_3 = f(x)$

4. **Groups and logic**
   A group is a structure $(G, 0, +)$ where G is a set, There is a special element $0 \in G$ called the identity and $+ : G \times G \to G$ is a binary operation such that the following properties are satisfied. For each property below, give an FOL sentence on the vocabulary $(+, =)$ such that it evaluates to true on G iff G satisfies the property.

   (a) The operation $+$ is associative.
   [*Hint: For any choice of x, y and z from G, $(x + y) + z = x + (y + z)$. Note that here $+$ is not necessarily the addition operation, $+$ only serves as a symbol, so the associativity property isn't inherently applicable to any such function. So you want your formula to be true iff the interpretation of $+$ has this property*].
   (b) The constant 0 is a right-identity for the operation $+$.
   [*Hint: For any choice of x, $x + 0 = x$*].
   (c) Every element in G has a right inverse.
   [*Hint: For each $x \in G$, we can find $y \in G$ such that $x + y = 0$*]
   (d) For any three elements $x, y, z \in G$, if $x + z = y + z$, then $x = y$

5. **Blood Relations**
   Consider a structure where the universe consists of people in a family network.
   The vocabulary is

$$\mathcal{V} = \{Parent, Female, Male, =\}$$

   - $Parent(x, y)$ is a binary predicate that evaluates to true iff$x$ is a parent of y
   - $Female(x)$ is a unary predicate that evaluates to true iff x is female
   - $Male(x)$ is a unary predicate that evaluates to true iff x is male
   - $=$: equality relation

   1. Write formulas defining the following relations.
      (a) Write a formula $Mother(m, c)$ with two free variables such that this formula evaluates to true on the above structure iff m is the mother of c.
      (b) Write a formula $Sibling(x, y)$ with two free variables such that this formula evaluates to true on the above structure iff x and y are siblings (they share at least one parent and $x \neq y$).

(c) Write a formula $Grandparent(g, c)$ with two free variables such that the formula evaluates to true iff g is a grandparent of c.

2. Write a sentence expressing "Every person has exactly one mother".
3. Define $Matriarch(m)$ as:
   - m is female
   - m has no parents

   Now write a sentence expressing:
   Every person is within two generations of at least one matriarch.

6. **Spot the difference**

   Consider directed graphs as structures over the vocabulary

$$\mathcal{V} = \{R(x, y), =\}$$

where $R(x, y)$ means there is a directed edge from x to y, and $=$ denotes equality.
Two directed graphs are given below.
Graph G_1
Vertices: $\{a, b, c, d\}$ Edges:

$$a \to b, \quad a \to c, \quad b \to c, \quad c \to d, \quad d \to a$$

Graph G_2
Vertices: $\{u, v, w, x\}$
Edges:

$$u \to v, \quad v \to w, \quad w \to x, \quad x \to u$$

Write a FOL sentence which distinguishes between these two graphs, i.e. one of them satisfies the sentence and other does not.

7. **Prefix Order on Binary Strings**

   Let $\mathcal{V} = \{\mathsf{Pref}, =\}$, where Pref is a binary predicate. Consider the $\mathcal{V}$-structure $\mathcal{B}$ whose universe is the set of all binary strings over $\{0, 1\}$, including the empty string ϵ. For strings u, v, the predicate $\mathsf{Pref}(u, v)$ is interpreted as:

$$\mathsf{Pref}(u, v) = \mathsf{true} \quad \text{iff } u \text{ is a prefix of } v.$$

As a warm-up, verify that the relation Pref on $\mathcal{B}$ is:

- *reflexive*, i.e. $\forall x\ \mathsf{Pref}(x, x)$,
- *transitive*, i.e.

$$\forall x \forall y \forall z\ \big((\mathsf{Pref}(x, y) \land \mathsf{Pref}(y, z)) \to \mathsf{Pref}(x, z)\big),$$

- *antisymmetric*, i.e.

$$\forall x \forall y\ \big((\mathsf{Pref}(x, y) \land \mathsf{Pref}(y, x)) \to x = y\big).$$

A binary relation satisfying the above three properties is called a *partially ordered set* (or *poset*).
A poset is called a *total order* if, in addition,

$$\forall x \forall y\ \big(\mathsf{Pref}(x, y) \lor \mathsf{Pref}(y, x)\big)$$

holds.

(a) Since the warm-up shows that Pref is a poset, what additional first-order condition over $\mathcal{V}$ must be imposed to ensure that Pref is *not* a total order?

(b) Write a formula $\mathsf{Imm}(x, y)$ over $\mathcal{V}$ such that $\mathsf{Imm}(x, y)$ holds iff y is an *immediate extension* of x; that is, x is a proper prefix of y, and there does not exist any string z such that x is a proper prefix of z and z is a proper prefix of y.

(c) Using the formula from part (b), write a sentence over $\mathcal{V}$ expressing that every string has exactly two immediate extensions.

8. **Bit-Vectors**

A bit vector of length n is a vector of n bits. An example bit-vector of length 3 is 101.

A bit-vector v of length n can be represented using a unary predicate S_v over the universe $\{0, 1, \dots n-1\}$. Specifically, $S_v(i) = \text{true}$ for $i \in \mathbb{N}$ iff the i^{th} bit of v is 1.

Now, consider the vocabulary $\mathcal{V} = \{S_x, S_y, <, =\}$, interpreted over the universe $\{0, 1, \dots n-1\}$, where:

- S_x, S_y are unary predicates whose interpretations define bit-vectors x and y of length n.
- $<$ is the standard "strictly less than" ordering of elements of $\{0, 1, \dots, n-1\}$.
- $=$ denotes equality of elements.

(a) Bitwise Equality

Write a FOL sentence on the above vocabulary that, when interpreted over the universe $\{0, 1, \dots n-1\}$ expresses:

"The bit-vector x and the bit-vector y are identical."

(b) Common Zeros

Similarly, write a FOL sentence that expresses:

"There is at least one position i where both vector x and vector y have a 0."

(c) Cardinality (The "Singleton" Property)

Similarly, write a FOL sentence that expresses: *"Vector x has exactly one bit set to 1."*

(d) Mind the gap

Write a FOL sentence that expresses the property that there is a 0 sandwiched between two 1s in bit-vector x, but there is no such sandwiched 0 in bit-vector y.