CS 228M : Logic for Computer Science Spring 2026

Practice Problem Set 5

Instructions:

- The following problems are meant for you to practice, so that your understanding of the topic improves.
- You must solve all problems to get the maximum benefit from practice problems.
- You must not submit your solutions to these problems. These are not going to be graded.
- A problem may have multiple solution techniques. Discussion among students is strongly encouraged in order to understand different perspectives.
- Questions marked * are comparably more difficult than the rest.

---

1. **School Mentorship** At a school, the following statements hold:
   1. Every teacher mentors at least one student.
   2. Every student who is mentored by a teacher passes an exam.
   3. Anyone who passes an exam is praised by someone.
   4. No one praises themselves.

   Using Natural Deduction, prove that if there is atleast one teacher at school, then there exist two distinct people such that one praises the other.

2. **Inexpressibility of Graph Connectedness in FOL** Let $\mathcal{L} = \{R\}$ be the language of directed graphs, where R is a binary relation symbol representing directed edges. We want to prove that "reachability" is not expressible in First-Order Logic (FOL). We will proceed by contradiction using the Compactness Theorem.

   Assume for the sake of contradiction that there exists a First-Order formula $\varphi(u, v)$ with exactly two free variables, u and v, such that for any graph G and nodes $a, b \in G$:

   $$G \models \varphi(a, b) \leftrightarrow \text{there is a directed path from } a \text{ to } b \text{ in } G$$

   Let us expand our language to $\mathcal{L}^* = \{R, c, d\}$, where c and d are new constant symbols.

   1. Write a First-Order sentence ψ_1 that states there is no path of length 1 (i.e., no direct edge) from c to d. Then, write a sentence ψ_2 stating there is no path of length 2 or less from c to d. Finally, generalize this: describe what the sentence ψ_n looks like for any integer $n \geq 1$.
   2. Consider the infinite set of sentences:

      $$\Gamma = \{\varphi(c, d)\} \cup \{\psi_n \mid n \geq 1\}$$

      Explain in plain English what it would mean for a single graph to satisfy the entire set Γ. Why is this intuitively impossible?
   3. Let Γ_0 be an arbitrary *finite* subset of Γ. Prove that Γ_0 is satisfiable.
      (Hint: Since Γ_0 is finite, there is some maximum index k for the sentences ψ_k contained in it. Construct a simple graph that satisfies both $\varphi(c, d)$ and all ψ_n up to ψ_k.)

4. Using your result from Part 3 and the Compactness Theorem, complete the proof to show that our initial assumption (the existence of $\varphi(u, v)$) must be false.

3. **Prove the sequents!** Prove the following sequents in first order logic using natural deduction:

   1. $\forall x(P(x) \to Q(x)), \forall x(Q(x) \to \neg R(x)), P(s), \forall x(T(x) \to R(x)) \vdash \neg T(s)$
   2. $\exists x(P(x) \land Q(x)), \forall x(Q(x) \to R(x)) \vdash \exists x(P(x) \land R(x))$
   3. $\forall x(P(x) \to Q), \exists x P(x) \vdash Q$
   4. $\forall x \forall y(R(x, y) \to \neg R(y, x)) \vdash \forall x \neg R(x, x)$
   5. $\neg \exists x P(x) \vdash \forall x \neg P(x)$
   6. $\forall x(P(x) \to \forall y Q(y)), P(a) \vdash \forall y Q(y)$
   7. $(\exists x P(x) \to Q) \vdash \forall x(P(x) \to Q)$
   8. $(\forall x P(x) \lor \forall x Q(x)) \vdash \forall x(P(x) \lor Q(x))$
   9. $\exists x \forall y R(x, y) \vdash \forall y \exists x R(x, y)$
   10. $\forall x(P(x) \leftrightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$

4. **FOL Modelling and Natural Deduction (Graphs)**

   Consider a domain of vertices of a directed graph. You are given the following statements:

   1. Every vertex has an outgoing edge to some vertex.
   2. If there is an edge from one vertex to another, then the second vertex is reachable from the first.
   3. Reachability is transitive.
   4. There is a vertex a in the graph.

   Use the following predicates:

   - $E(x, y)$: there is a directed edge from x to y
   - $R(x, y)$: y is reachable from x

   (a) Translate the first three statements into first-order logic. (Hint: Use quantifiers wisely, and pay attention to their order)

   (b) Using natural deduction, prove:

   $$\exists y \, Reach(a, y)$$

   (c) Can we conclude:

   $$\exists y \, (y \neq a \land Reach(a, y))$$

   holds for all graphs satisfying the 4 statements at the beginning of the question? Justify your answer.

5. **Distributivity of Quantifiers over Logical Connectives**

   Let $\varphi(x)$ and $\psi(x)$ be first-order formulas.

   For each of the following, determine whether the equivalence is logically valid:

   1. $\forall x \, (\varphi(x) \land \psi(x)) \;\leftrightarrow\; (\forall x \, \varphi(x)) \land (\forall x \, \psi(x))$
   2. $\exists x \, (\varphi(x) \lor \psi(x)) \;\leftrightarrow\; (\exists x \, \varphi(x)) \lor (\exists x \, \psi(x))$
   3. $\forall x \, (\varphi(x) \lor \psi(x)) \;\leftrightarrow\; (\forall x \, \varphi(x)) \lor (\forall x \, \psi(x))$
   4. $\exists x \, (\varphi(x) \land \psi(x)) \;\leftrightarrow\; (\exists x \, \varphi(x)) \land (\exists x \, \psi(x))$

   - If valid, prove using natural deduction.
   - If invalid, give a counterexample structure.
   - For invalid cases, check whether at least one direction holds.

6. **Primary Keys, Foreign Keys and First-Order Logic**

A university stores data using the following tables:

$$\texttt{Student}(sid, name), \qquad \texttt{Course}(cid, title), \qquad \texttt{Registered}(rid, sid, cid)$$

Here, *sid* is a student ID, *cid* is a course ID, and *rid* is a registration ID.

A *primary key* is a field whose value uniquely identifies a row in a table, i.e. no two rows can have the same value for the primary key.

Thus:

- *sid* is the primary key of the `Student` table,
- *cid* is the primary key of the `Course` table,
- *rid* is the primary key of the `Registered` table.

A *foreign key* is a field in one table whose value must already appear as a primary key in another table.

In $\texttt{Registered}(rid, sid, cid)$:

- the value of *sid* should be a valid student ID, and
- the value of *cid* should be a valid course ID.

So the fact

$$\texttt{Registered}(r1, s1, c1)$$

is reasonable only if $s1$ is a valid student ID and $c1$ is a valid course ID.

We use the following predicates:

- $Student(x)$: x is a valid student ID
- $Course(x)$: x is a valid course ID
- $StudentName(x, n)$: student ID x has name n
- $Registered(r, x, y)$: registration ID r records that student ID x is registered in course ID y

(a) Write a first-order logic formula expressing the *primary key property* for student IDs: if a student ID is associated with two names, then those names must be equal.

(b) Write *two* first-order logic formulas expressing the *foreign key property* for registrations:

1. if $Registered(r, x, y)$, then x must be a valid student ID;
2. if $Registered(r, x, y)$, then y must be a valid course ID.

(c) Suppose the `Course` table is now

$$\texttt{Course}(cid, title, slot),$$

where *slot* denotes the time-slot in which the course is offered.

Using a predicate $CourseSlot(c, s)$, where $CourseSlot(c, s)$ means that course ID c is offered in slot s, write a first-order logic formula expressing the following constraint: *No student can be registered in two* different *courses in the same slot.*

7. **A Set-Theoretic Proof**

In this question, we work with a first-order language for sets having:

- a binary relation symbol $\in$,
- a binary relation symbol $\subseteq$,
- a binary function symbol $-$ for set difference.

Assume the following axioms:

$$\Sigma = \left\{ \begin{array}{l} \forall x \forall y \Big(\forall z \, (z \in x \leftrightarrow z \in y) \to x = y \Big), \\ \forall x \forall y \Big(x \subseteq y \leftrightarrow \forall z \, (z \in x \to z \in y) \Big), \\ \forall x \forall y \forall z \Big(z \in (x - y) \leftrightarrow (z \in x \wedge z \notin y) \Big) \end{array} \right\}.$$

These axioms say the following:

- **Extensionality:** if two sets have exactly the same elements, then they are equal.
- **Definition of subset:** $x \subseteq y$ means that every element of x is also an element of y.
- **Definition of set difference:** an element belongs to $x - y$ exactly when it belongs to x and does not belong to y.

Prove that

$$\Sigma \vdash \forall x \forall y \Big(x \subseteq y \to \exists z \, (y - z = x) \Big).$$

8. **Internship season in FOL!**

At the college placement office, internship applications are processed before interview calls are sent out.

The following predicates are used:

- $Complete(x)$: x is a complete application.
- $OnTime(x)$: x is submitted on time.
- $Screened(x)$: x is screened by the placement office.
- $Shortlisted(x)$: x is shortlisted.
- $ManualReview(x)$: x is sent to manual review.
- $InterviewInvite(x)$: x gets an interview invitation.

The domain is the set of all internship applications.

Translate the following English statements into first-order logic:

1. Every complete and on-time application is screened.
2. Every screened application is either shortlisted or sent to manual review.
3. No application sent to manual review is on time.
4. Every shortlisted application gets an interview invitation.

Then, using the formulas above as premises, prove:

$$\forall x \big((Complete(x) \wedge OnTime(x)) \to InterviewInvite(x) \big).$$