

Practice Problem Set 5 Solutions

1. School Mentorship

Solution: To solve this problem using natural deduction, we first convert all the aforementioned rules into first order logic.

Defining the predicates:

- **Domain:** All people at the school.
- $T(x)$: x is a teacher.
- $S(x)$: x is a student.
- $M(x, y)$: x mentors y .
- $E(x)$: x passes an exam.
- $P(x, y)$: x praises y .

Using these predicates the conditions can be rewritten as follows:

- **Premise 1:** $\forall x(T(x) \rightarrow \exists y(S(y) \wedge M(x, y)))$
- **Premise 2:** $\forall y((S(y) \wedge \exists x(T(x) \wedge M(x, y))) \rightarrow E(y))$
- **Premise 3:** $\forall y(E(y) \rightarrow \exists zP(z, y))$
- **Premise 4:** $\forall x\neg P(x, x)$
- **Conclusion:** $\exists xT(x) \rightarrow \exists x\exists y(P(x, y) \wedge \neg(x = y))$

Proof:

1.	$\forall x(T(x) \rightarrow \exists y(S(y) \wedge M(x, y)))$	premise
2.	$\forall y((S(y) \wedge \exists x(T(x) \wedge M(x, y))) \rightarrow E(y))$	premise
3.	$\forall y(E(y) \rightarrow \exists zP(z, y))$	premise
4.	$\forall x\neg P(x, x)$	premise
5.	$\exists xT(x)$	assumption
6.	$T(a)$	assumption
7.	$T(a) \rightarrow \exists y(S(y) \wedge M(a, y))$	$\forall E$ 1
8.	$\exists y(S(y) \wedge M(a, y))$	$\rightarrow E$ 7, 6
9.	$S(b) \wedge M(a, b)$	assumption
10.	$(S(b) \wedge \exists x(T(x) \wedge M(x, b))) \rightarrow E(b)$	$\forall E$ 2
11.	$S(b)$	$\wedge E$ 9
12.	$M(a, b)$	$\wedge E$ 9
13.	$T(a) \wedge M(a, b)$	$\wedge I$ 6, 12
14.	$\exists x(T(x) \wedge M(x, b))$	$\exists I$ 13
15.	$S(b) \wedge \exists x(T(x) \wedge M(x, b))$	$\wedge I$ 11, 14
16.	$E(b)$	$\rightarrow E$ 10, 15
17.	$E(b) \rightarrow \exists zP(z, b)$	$\forall E$ 3
18.	$\exists zP(z, b)$	$\rightarrow E$ 17, 16
19.	$P(c, b)$	assumption
20.	$c = b$	assumption
21.	$P(b, b)$	$=E$ 19, 20
22.	$\neg P(b, b)$	$\forall E$ 4
23.	\perp	$\neg E$ 21, 22
24.		
25.	$\neg(c = b)$	$\neg I$ 20–23
26.	$P(c, b) \wedge \neg(c = b)$	$\wedge I$ 19, 24
27.	$\exists y(P(c, y) \wedge \neg(c = y))$	$\exists I$ 25
28.	$\exists x\exists y(P(x, y) \wedge \neg(x = y))$	$\exists I$ 26
29.		
30.	$\exists x\exists y(P(x, y) \wedge \neg(x = y))$	$\exists E$ 18, 19–27
31.		
32.	$\exists x\exists y(P(x, y) \wedge \neg(x = y))$	$\exists E$ 8, 9–28
33.		
34.	$\exists x\exists y(P(x, y) \wedge \neg(x = y))$	$\exists E$ 5, 6–29
35.		
36.	$\exists xT(x) \rightarrow \exists x\exists y(P(x, y) \wedge \neg(x = y))$	$\rightarrow I$ 5–30

2. Inexpressibility of Graph Connectedness in FOL

Solution:

1.
 - $\psi_1 \equiv \neg R(c, d)$
 - $\psi_2 \equiv \neg R(c, d) \wedge \neg \exists x (R(c, x) \wedge R(x, d))$
 - ψ_n states there is no path of length $\leq n$. It can be written as the conjunction of ψ_{n-1} and the formula stating there is no path of length exactly n :

$$\neg \exists x_1 \dots \exists x_{n-1} (R(c, x_1) \wedge R(x_1, x_2) \wedge \dots \wedge R(x_{n-1}, d))$$

2. If a graph satisfies Γ , it must satisfy $\varphi(c, d)$ (meaning there *is* a path from c to d of some finite length) AND it must satisfy every ψ_n (meaning there is no path of length 1, no path of length 2, no path of length 3, and so on infinitely). This is intuitively impossible because any actual path must have a finite length k , which would immediately violate the sentence ψ_k .
3. Because Γ_0 is finite, it can only contain a finite number of the “no short path” sentences. Let k be the largest integer such that $\psi_k \in \Gamma_0$. To show Γ_0 is satisfiable, we construct a model for it.

Create a directed path graph with $k + 2$ vertices: $v_0 \rightarrow v_1 \rightarrow v_2 \dots \rightarrow v_{k+1}$. Let the constant c map to v_0 and d map to v_{k+1} . Because there is a path from c to d , the graph satisfies $\varphi(c, d)$. Because the shortest (and only) path has length $k + 1$, it satisfies all ψ_n for $n \leq k$. Thus, Γ_0 is satisfied.

4. The Compactness Theorem states that if every finite subset of a set of sentences is satisfiable, then the entire infinite set is satisfiable. By Part 3, every finite subset Γ_0 is satisfiable. Therefore, the infinite set Γ must have a model M .

However, as established in Part 2, a model satisfying Γ must simultaneously have a finite path between c and d (to satisfy $\varphi(c, d)$) and have no path of any finite length between c and d (to satisfy all ψ_n). This is a direct logical contradiction.

Therefore, our assumption that $\varphi(u, v)$ exists must be false. Reachability cannot be expressed in First-Order Logic.

3. Prove the sequents!**Solution:**

1.	$\forall x(P(x) \rightarrow Q(x))$	premise
2.	$\forall x(Q(x) \rightarrow \neg R(x))$	premise
3.	$P(s)$	premise
4.	$\forall x(T(x) \rightarrow R(x))$	premise
5.	$T(s)$	assumption
6.	$T(s) \rightarrow R(s)$	$\forall E$ 4
7.	$R(s)$	$\rightarrow E$ 6, 5
1. 8.	$P(s) \rightarrow Q(s)$	$\forall E$ 1
9.	$Q(s)$	$\rightarrow E$ 8, 3
10.	$Q(s) \rightarrow \neg R(s)$	$\forall E$ 2
11.	$\neg R(s)$	$\rightarrow E$ 10, 9
12.	\perp	$\neg E$ 7, 11
13.		
14.	$\neg T(s)$	$\neg I$ 5–12

1.	$\exists x(P(x) \wedge Q(x))$	premise
2.	$\forall x(Q(x) \rightarrow R(x))$	premise
3.	$P(a) \wedge Q(a)$	assumption
4.	$Q(a) \rightarrow R(a)$	$\forall E$ 2
5.	$Q(a)$	$\wedge E$ 3
2. 6.	$R(a)$	$\rightarrow E$ 4, 5
7.	$P(a)$	$\wedge E$ 3
8.	$P(a) \wedge R(a)$	$\wedge I$ 7, 6
9.	$\exists x(P(x) \wedge R(x))$	$\exists I$ 8
10.		
11.	$\exists x(P(x) \wedge R(x))$	$\exists E$ 1, 3–9

1.	$\forall x(P(x) \rightarrow Q)$	premise
2.	$\exists xP(x)$	premise
3.	$P(a)$	assumption
3. 4.	$P(a) \rightarrow Q$	$\forall E$ 1
5.	Q	$\rightarrow E$ 4, 3
6.		
7.	Q	$\exists E$ 2, 3–5

1.	$\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$	premise
2.	a	assumption
3.	$R(a, a)$	assumption
4.	$\forall y (R(a, y) \rightarrow \neg R(y, a))$	$\forall E$ 1
5.	$R(a, a) \rightarrow \neg R(a, a)$	$\forall E$ 4
4. 6.	$\neg R(a, a)$	$\rightarrow E$ 5, 3
7.	\perp	$\neg E$ 3, 6
8.		
9.	$\neg R(a, a)$	$\neg I$ 3-7
10.		
11.	$\forall x \neg R(x, x)$	$\forall I$ 2-8

1.	$\neg \exists x P(x)$	premise
2.	a	assumption
3.	$P(a)$	assumption
4.	$\exists x P(x)$	$\exists I$ 3
5. 5.	\perp	$\neg E$ 4, 1
6.		
7.	$\neg P(a)$	$\neg I$ 3-5
8.		
9.	$\forall x \neg P(x)$	$\forall I$ 2-6

1.	$\forall x (P(x) \rightarrow \forall y Q(y))$	premise
2.	$P(a)$	premise
6. 3.	$P(a) \rightarrow \forall y Q(y)$	$\forall E$ 1
4.	$\forall y Q(y)$	$\rightarrow E$ 3, 2

1.	$\exists x P(x) \rightarrow Q$	premise
2.	a	assumption
3.	$P(a)$	assumption
4.	$\exists x P(x)$	$\exists I$ 3
7. 5.	Q	$\rightarrow E$ 1, 4
6.		
7.	$P(a) \rightarrow Q$	$\rightarrow I$ 3-5
8.		
9.	$\forall x (P(x) \rightarrow Q)$	$\forall I$ 2-6

1.	$\forall xP(x) \vee \forall xQ(x)$	premise
2.	a	assumption
3.	$\forall xP(x)$	assumption
4.	$P(a)$	$\forall E$ 3
5.	$P(a) \vee Q(a)$	$\vee I$ 4
6.		
8.	$\forall xQ(x)$	assumption
8.	$Q(a)$	$\forall E$ 6
9.	$P(a) \vee Q(a)$	$\vee I$ 7
10.		
11.	$P(a) \vee Q(a)$	$\vee E$ 1, 3–5, 6–8
12.		
13.	$\forall x(P(x) \vee Q(x))$	$\forall I$ 2–9
1.	$\exists x\forall yR(x, y)$	premise
2.	b	assumption
3.	$\forall yR(a, y)$	assumption
4.	$R(a, b)$	$\forall E$ 3
9.	$\exists xR(x, b)$	$\exists I$ 4
6.		
7.	$\exists xR(x, b)$	$\exists E$ 1, 3–5
8.		
9.	$\forall y\exists xR(x, y)$	$\forall I$ 2–6
1.	$\forall x(P(x) \leftrightarrow Q(x))$	premise
2.	$\exists xP(x)$	premise
3.	$P(a)$	assumption
4.	$P(a) \leftrightarrow Q(a)$	$\forall E$ 1
10.	$Q(a)$	$\leftrightarrow E$ 4, 3
6.	$\exists xQ(x)$	$\exists I$ 5
7.		
8.	$\exists xQ(x)$	$\exists E$ 2, 3–6

4. FOL Modelling and Natural Deduction (Graphs)

Solution:

(a) FOL Modelling Part

- (1) $\forall x \exists y \text{Edge}(x, y)$
- (2) $\forall x \forall y (\text{Edge}(x, y) \rightarrow \text{Reach}(x, y))$
- (3) $\forall x \forall y \forall z ((\text{Reach}(x, y) \wedge \text{Reach}(y, z)) \rightarrow \text{Reach}(x, z))$

—
(b) Natural Deduction Proof

We prove $\exists y \text{ Reach}(a, y)$.

1. $\forall x \exists y \text{ Edge}(x, y)$ (Premise)
2. $\forall x \forall y (\text{Edge}(x, y) \rightarrow \text{Reach}(x, y))$ (Premise)
3. $\exists y \text{ Edge}(a, y)$ (\forall -elimination on 1)
4. Assume $\text{Edge}(a, b)$ for some fresh b (\exists -elimination)
5. $\text{Edge}(a, b) \rightarrow \text{Reach}(a, b)$ (\forall -elimination on 2)
6. $\text{Reach}(a, b)$ (Modus Ponens from 4,5)
7. $\exists y \text{ Reach}(a, y)$ (\exists -introduction)

—
(c) Can we conclude $\exists y(y \neq a \wedge \text{Reach}(a, y))$?

No, this does not necessarily follow.

Counterexample: Consider a graph with a single vertex a and a self-loop:

$$\text{Edge}(a, a)$$

Then:

- $\forall x \exists y \text{ Edge}(x, y)$ is satisfied
- $\text{Reach}(a, a)$ holds
- But there is no $y \neq a$

Hence,

$$\exists y(y \neq a \wedge \text{Reach}(a, y))$$

is not guaranteed.

5. Distributivity of Quantifiers over Logical Connectives

Solution:

1. $\forall x(\varphi \wedge \psi) \leftrightarrow (\forall x\varphi) \wedge (\forall x\psi)$

Valid

(\Rightarrow):

1. $\forall x(\varphi(x) \wedge \psi(x))$
2. $\varphi(a) \wedge \psi(a)$ (\forall -elim)
3. $\varphi(a), \psi(a)$ (\wedge -elim)
4. $\forall x\varphi(x), \forall x\psi(x)$ (\forall -intro)

(\Leftarrow):

1. $\forall x\varphi(x), \forall x\psi(x)$
2. $\varphi(a), \psi(a)$ (\forall -elim)
3. $\varphi(a) \wedge \psi(a)$ (\wedge -intro)
4. $\forall x(\varphi(x) \wedge \psi(x))$ (\forall -intro)

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*2. $\exists x(\varphi \vee \psi) \leftrightarrow (\exists x\varphi) \vee (\exists x\psi)$

Valid

(\Rightarrow):

1. $\exists x(\varphi(x) \vee \psi(x))$
2. Assume $\varphi(a) \vee \psi(a)$ (\exists -elim)
3. Case 1: $\varphi(a) \Rightarrow \exists x\varphi(x)$
4. Case 2: $\psi(a) \Rightarrow \exists x\psi(x)$
5. $(\exists x\varphi(x)) \vee (\exists x\psi(x))$

(\Leftarrow):

1. $(\exists x\varphi(x)) \vee (\exists x\psi(x))$
2. Case 1: $\exists x\varphi(x) \Rightarrow \exists x(\varphi(x) \vee \psi(x))$
3. Case 2: $\exists x\psi(x) \Rightarrow \exists x(\varphi(x) \vee \psi(x))$
4. $\exists x(\varphi(x) \vee \psi(x))$

—

3. $\forall x(\varphi \vee \psi) \leftrightarrow (\forall x\varphi) \vee (\forall x\psi)$

Invalid

Counterexample:

Let the domain be $\{1, 2\}$:

$$\begin{aligned} \varphi(1) &= \text{true}, & \varphi(2) &= \text{false} \\ \psi(1) &= \text{false}, & \psi(2) &= \text{true} \end{aligned}$$

Then:

$$\forall x(\varphi(x) \vee \psi(x)) \text{ is true}$$

but

$$(\forall x\varphi(x)) \vee (\forall x\psi(x)) \text{ is false}$$

However:

$$(\forall x\varphi(x)) \vee (\forall x\psi(x)) \Rightarrow \forall x(\varphi(x) \vee \psi(x))$$

is valid.

—

4. $\exists x(\varphi \wedge \psi) \leftrightarrow (\exists x\varphi) \wedge (\exists x\psi)$

Invalid

Counterexample:

Let the domain be $\{1, 2\}$:

$$\varphi(1) = \text{true}, \quad \psi(1) = \text{false}$$

$$\varphi(2) = \text{false}, \quad \psi(2) = \text{true}$$

Then:

$$(\exists x\varphi(x)) \wedge (\exists x\psi(x)) \text{ is true}$$

but

$$\exists x(\varphi(x) \wedge \psi(x)) \text{ is false}$$

However:

$$\exists x(\varphi(x) \wedge \psi(x)) \Rightarrow (\exists x\varphi(x)) \wedge (\exists x\psi(x))$$

is valid.

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Summary

- Valid equivalences: (1), (2)
- Invalid equivalences: (3), (4)
- In invalid cases, only one direction holds

6. Primary Keys, Foreign Keys and First-Order Logic

Solution:

(a) The required formula is:

$$\forall x\forall n_1\forall n_2 \left((StudentName(x, n_1) \wedge StudentName(x, n_2)) \rightarrow n_1 = n_2 \right)$$

This says that if the same student ID x is associated with two names n_1 and n_2 , then those two names must in fact be equal.

(b) The two required formulas are:

$$\forall r\forall x\forall y (Registered(r, x, y) \rightarrow Student(x))$$

and

$$\forall r\forall x\forall y (Registered(r, x, y) \rightarrow Course(y))$$

The first formula says that every student ID appearing in a registration record must be a valid student ID, and the second says that every course ID appearing in a registration record must be a valid course ID.

(c) The required formula is:

$$\begin{aligned} \forall x\forall r_1\forall r_2\forall c_1\forall c_2\forall s \left((Registered(r_1, x, c_1) \wedge Registered(r_2, x, c_2)) \right. \\ \left. \wedge CourseSlot(c_1, s) \wedge CourseSlot(c_2, s) \right) \\ \rightarrow c_1 = c_2 \end{aligned}$$

This says that if a student x is registered in two courses c_1 and c_2 , and both courses are offered in the same slot s , then those two courses must actually be the same course. Hence, a student cannot be registered in two different courses in the same slot.

7. A Set-Theoretic Proof

Solution: Let x, y be arbitrary and assume

$$x \subseteq y.$$

We must show

$$\exists z (y - z = x).$$

Take

$$z := y - x.$$

It is enough to prove

$$y - (y - x) = x.$$

By extensionality, it suffices to show

$$\forall w (w \in y - (y - x) \leftrightarrow w \in x).$$

So let w be arbitrary.

(\Rightarrow) Assume

$$w \in y - (y - x).$$

By the axiom for set difference,

$$w \in y \wedge w \notin (y - x).$$

Hence $w \in y$. If $w \notin x$, then again by the axiom for set difference,

$$w \in y - x,$$

contradicting $w \notin (y - x)$. Therefore $w \in x$.

(\Leftarrow) Assume

$$w \in x.$$

Since $x \subseteq y$, by the subset axiom we get

$$\forall u (u \in x \rightarrow u \in y),$$

and hence $w \in y$.

Now suppose

$$w \in y - x.$$

Then by the axiom for set difference,

$$w \in y \wedge w \notin x,$$

which contradicts $w \in x$. So $w \notin (y - x)$.

Therefore,

$$w \in y \wedge w \notin (y - x),$$

and hence, by the axiom for set difference again,

$$w \in y - (y - x).$$

Thus

$$w \in y - (y - x) \leftrightarrow w \in x.$$

Since w was arbitrary,

$$\forall w (w \in y - (y - x) \leftrightarrow w \in x).$$

By extensionality,

$$y - (y - x) = x.$$

Therefore,

$$\exists z (y - z = x).$$

Since x, y were arbitrary, we conclude

$$\forall x \forall y (x \subseteq y \rightarrow \exists z (y - z = x)).$$

Hence,

$$\Sigma \vdash \forall x \forall y (x \subseteq y \rightarrow \exists z (y - z = x)).$$

8. Internship season in FOL!

Solution:

- (1) $\forall x ((Complete(x) \wedge OnTime(x)) \rightarrow Screened(x)),$
- (2) $\forall x (Screened(x) \rightarrow (Shortlisted(x) \vee ManualReview(x))),$
- (3) $\forall x (ManualReview(x) \rightarrow \neg OnTime(x)),$
- (4) $\forall x (Shortlisted(x) \rightarrow InterviewInvite(x)).$

To prove

$$\forall x ((Complete(x) \wedge OnTime(x)) \rightarrow InterviewInvite(x)),$$

we take an arbitrary application a and show

$$(Complete(a) \wedge OnTime(a)) \rightarrow InterviewInvite(a).$$

The proof proceeds by implication introduction and then by case analysis on the disjunction obtained from screening.

Let

$$C(x) := Complete(x), \quad O(x) := OnTime(x), \quad S(x) := Screened(x),$$

$$L(x) := Shortlisted(x), \quad M(x) := ManualReview(x), \quad I(x) := InterviewInvite(x).$$

Then the premises are:

- (1) $\forall x (C(x) \wedge O(x) \rightarrow S(x))$
- (2) $\forall x (S(x) \rightarrow (L(x) \vee M(x)))$
- (3) $\forall x (M(x) \rightarrow \neg O(x))$
- (4) $\forall x (L(x) \rightarrow I(x))$

We prove:

$$\forall x (C(x) \wedge O(x) \rightarrow I(x)).$$

- | | | |
|-----|---|--------------------------|
| 1. | $\forall x (C(x) \wedge O(x) \rightarrow S(x))$ | premise |
| 2. | $\forall x (S(x) \rightarrow (L(x) \vee M(x)))$ | premise |
| 3. | $\forall x (M(x) \rightarrow \neg O(x))$ | premise |
| 4. | $\forall x (L(x) \rightarrow I(x))$ | premise |
| 5. | $C(a) \wedge O(a)$ | assumption |
| 6. | $C(a) \wedge O(a) \rightarrow S(a)$ | $\forall E$ 1 |
| 7. | $S(a)$ | $\rightarrow E$ 5, 6 |
| 8. | $S(a) \rightarrow (L(a) \vee M(a))$ | $\forall E$ 2 |
| 9. | $L(a) \vee M(a)$ | $\rightarrow E$ 7, 8 |
| 10. | $L(a)$ | assumption |
| 11. | $L(a) \rightarrow I(a)$ | $\forall E$ 4 |
| 12. | $I(a)$ | $\rightarrow E$ 10, 11 |
| 13. | | |
| 14. | $M(a)$ | assumption |
| 15. | $M(a) \rightarrow \neg O(a)$ | $\forall E$ 3 |
| 16. | $\neg O(a)$ | $\rightarrow E$ 13, 14 |
| 17. | $O(a)$ | $\wedge E$ 5 |
| 18. | \perp | $\neg E$ 15, 16 |
| 19. | $I(a)$ | $\perp E$ 17 |
| 20. | | |
| 21. | $I(a)$ | $\vee E$ 9, 10–12, 13–18 |
| 22. | | |
| 23. | $C(a) \wedge O(a) \rightarrow I(a)$ | $\rightarrow I$ 5–19 |
| 24. | $\forall x (C(x) \wedge O(x) \rightarrow I(x))$ | $\forall I$ |