Roll No.: ______________________

CS 228M: Logic in Computer Science Spring 2026

Quiz 2

Time: 8:30 - 9:30 *Max Marks: 20*

Instructions:

- ***Please write your roll number on all pages in the space provided at the top.***
- *Be brief, complete, and stick to what has been asked.*
- ***You must write your answer for every question only in the space allocated for answering the question. Answers written outside the allocated space risk not being graded.***
- ***You can use an extra answer book for rough calculations. You must write your roll number on the extra answer book if you are using one.***
- ***You must submit this question+answer book in its entirey along with any extra answer book for rough calculations (if you used one).***
- *Untidy presentation of answers, and random ramblings will be penalized by negative marks.*
- *Unless asked for explicitly, you may cite results/proofs covered in class without reproducing them.*
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others. All detected cases of copying will be reported to DADAC with names and roll nos. of all involved. The stakes are high if you get reported to DADAC, so you are strongly advised not to risk this.***

DO NOT TURN THIS PAGE UNTIL YOU ARE ASKED TO.

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1. **Quantified Boolean Formulas** [10]

A quantified Boolean formula (QBF) is obtained by universally/existentially quantifying (a subset of) variables in a propositional formula. For example, $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$ is a propositional formula, and $\forall x_1 \big(\exists x_2 \big((x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)\big)\big)$ is a QBF.

Given that propositional variables can only take one of two values, one may view $\exists x\, \varphi(x)$ as shorthand for $\varphi[x \mapsto \top] \vee \varphi[x \mapsto \bot]$, and similarly $\forall x\, \varphi(x)$ as shorthand for $\varphi[x \mapsto \top] \wedge \varphi[x \mapsto \bot]$. Here, $\varphi[x \mapsto \psi]$ denotes the formula obtained by replacing all occurrences of x in φ by ψ. Note that if we expand all shorthand notations in this manner, the size of the resulting propositional formula can be 2^n times the size of the QBF, if the QBF has n quantified variables. Thus, although QBFs are no more expressive than propositional formulas, they can be exponentially more succinct.

Note: QBFs don't strictly follow the syntax of first order logic (FOL), since in FOL, we quantify variables that take values from a universe, while propositional operators ($\wedge, \vee, \neg$) are applied to predicates (not variables).

(a) Prove using (a) the expansion-shorthand view of quantifiers given above, and (b) natural deduction proof rules of propositional logic, the following: [5]

$$\exists x \big(\forall y\, \varphi(x, y, z)\big) \;\rightarrow\; \forall y \big(\exists x\, \varphi(x, y, z)\big),$$

where $\varphi(x, y, z)$ is a propositional logic formula over x, y, z.

To get marks, you must show your steps clearly, number the steps, and label each step either by "$\forall$ expansion", "$\exists$ expansion", or the usual way in which natural deduction proof steps are labeled (premise, assumption, or proof rule with previous step numbers corresponding to premises). You can label the derivation of $\varphi[x \mapsto \top] \vee \varphi[x \mapsto \bot]$ from $\exists x\, \varphi(x)$, and also the derivation of $\exists x\, \varphi(x)$ from $\varphi[x \mapsto \top] \vee \varphi[x \mapsto \bot]$ by "$\exists$ expansion"; similarly for "$\forall$ expansion".

(b) Consider the following QBF: 5

$\exists y_1 \exists y_2 \forall x_1 \forall x_2 \forall x_3\, (x_1 \vee x_2 \vee x_4 \vee \neg y_2) \wedge (\neg x_2 \vee \neg x_3 \vee \neg y_3 \vee y_2) \wedge (x_3 \vee \neg x_4 \vee \neg y_1 \vee y_3)$

It is clear that after expansion of quantifiers, this becomes a propositional formula on variables x_4 and y_3. Let's call this $\alpha(x_4, y_3)$.

1. *[2.5]* Without expanding the given formula, give values of x_4 and y_3 that make $\alpha(x_4, y_3)$ evaluate to 1, i.e. true. Give justification for your answer.

2. *[2.5]* Without expanding the given formula, give values of x_4 and y_3 that make $\alpha(x_4, y_3)$ evaluate to 0, i.e. false. Give justificaiton for your answer.

2. **First-order properties of numbers** [10]

For purposes of this question, let the vocabulary $\mathcal{V}$ for first-order logic formulas be $\{+, \otimes, =, <\}$, where $+$ is a binary function symbol, $\otimes$ is a unary function symbol, and $<$ is a binary predicate symbol. The $\mathcal{V}$-structure M under consideration has $U^M = \mathbb{N} = \{0, 1, 2, \ldots\}$ as universe. The function symbol $+$ is interpreted as addition of natural numbers, and the predicate symbol $<$ is interpreted as "strictly less than" in the ordering of natural numbers, viz. $<^M (1, 2) = \textsf{true}$ but $<^M (2, 1) = \textsf{false}$. The interpretation $\otimes^M$ of the unary function symbol $\otimes$ is not disclosed to you for now.

(a) Give a $\mathcal{V}$-sentence ψ that expresses exactly the following property of natural numbers, when interpreted over M. In other words, you must give a sentence ψ that evaluates to $\textsf{true}$ if and only if the natural numbers have the following property, where the interpretaton of function and predicate symbols is as in M: [4]

There exist infinitely many natural numbers x ***such that*** $\otimes(y)$ ***is strictly greater than*** x ***for all natural numbers*** y ***strictly greater than*** x.

(b) Write a formula $\varphi(x)$ with free variable x such that $\varphi(x)$ evaluates to `true` on M for every binding $\alpha : x \mapsto n$, where n is a natural number with the following property: [4]

$\otimes^M(m)$ is greater than m for every natural number m greater than n.

For all other bindings of x, $\varphi(x)$ must evaluate to `false` when evaluated on M.

(c) Suppose you are now told that the interpretation $\otimes^M$ is the squaring operation, i.e. $\otimes^M(m) = m^2$ for every $m \in \mathbb{N}$. Use your answer to the previous sub-question to give a natural number n such that if $\alpha(x) = n$, then $M, \alpha \models \varphi(x)$. Show why your formula evaluates to true for this binding of x. [2]

EMPTY PAGE: MAY BE USED FOR ROUGH WORK