Tutorial 2

Instructions:

- The following problems are meant for you to practice, so that your understanding of the topic improves.
- You must solve all problems to get the maximum benefit from practice problems.
- You must not submit your solutions to these problems. These are not going to be graded.
- A problem may have multiple solution techniques. Discussion among students is strongly encouraged in order to understand different perspectives.

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1. (a) Let φ be the CNF formula $(x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_4 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$. Using only the resolution rule (and nothing else), show that φ is unsatisfiable. You must give your answer as a resolution refutation tree, i.e. a tree that shows which clauses are used to resolve to obtain what resolvent, until the empty clause is obtained as a resolvent. 10

(b) Let α be an *unsatisfiable* CNF formula with m clauses over n variables, where the minimum number of literals in a clause is r, and $m, n \geq 1,\ r \geq 2$. As an example, for the formula in part (a) of this problem, $m = 7$, $n = 4$ and $r = 2$. Show using resolution-based reasoning that for every such formula α, there exists a CNF formula β with the following properties: 20

- β is *unsatisfiable*
- β has $m + 1$ clauses, $n + 1$ variables
- α and β share $m - 1$ clauses
- the minimum number of literals in a clause of β is r (i.e. same as that of α).

*[**Aside:** A correct answer to this problem gives a way to construct non-trivial unsatisfiable CNF formulas on n variables, such that the size of the formula grows at most linearly with n.]*

2. Consider the formula $\varphi \equiv x_1 \vee \neg(x_2 \wedge \neg(x_3 \leftrightarrow (x_1 \rightarrow \neg x_2)))$.

   (a) Give a DNF formula that is semantically equivalent to ϕ, using techniques studied in class today. 10

(b) Use Tseitin encoding to write an equisatisfiable formula ψ in conjunctive normal form. Would $\neg\psi$ be also equisatisfiable with $\neg\varphi$? Give justification for your answer. 20

3. Consider the parity function, $\mathsf{PARITY} : \{0,1\}^n \mapsto \{0,1\}$, where PARITY evaluates to 1 if and only if an odd number of inputs is 1. In all of the CNFs below, we assume that each clause contains any variable at most once, i.e. no clause contains expressions of the form $p \wedge \neg p$ or $p \vee \neg p$. Furthermore, all clauses are assumed to be distinct.

   (a) Prove that any CNF representation of PARITY must have n literals (from distinct variables) in every clause. 15

   [Hint: Take a clause, and suppose the variable v is missing from it. What happens when you flip v?]

(b) Prove that any CNF representation of PARITY must have $\geq 2^{n-1}$ clauses. 15
[Hint: What is the relation between CNF/DNF and truth tables?]