CS 228M : Logic for Computer Science Spring 2026

Tutorial 3

Instructions:

- The following problems are meant for you to practice, so that your understanding of the topic improves.
- You must solve all problems to get the maximum benefit from practice problems.
- You must not submit your solutions to these problems. These are not going to be graded.
- A problem may have multiple solution techniques. Discussion among students is strongly encouraged in order to understand different perspectives.

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1. **Naturally deduced**

   (a) Consider the following natural deduction (ND) "proof" of validity of the propositional logic formula $x_1 \to x_2$.
   You are told that at least one line of the proof is incorrect. You are required to (i) find the first line (in ascending order of line numbers) of the proof that is incorrect, (ii) indicate why the line is incorrect, and (iii) indicate what a correct application of a proof rule (not a derived rule) at that line would have yielded.

1.	$\neg(\neg x_1 \vee x_2)$	assumption
2.	$\neg x_1$	assumption
3.	$\neg x_1 \vee x_2$	$\vee_{i1}$ 2
4.	$\bot$	$\neg_e$ 1, 3
5.	x_2	$\bot_e$ 4
6.	$\neg x_1 \vee x_2$	$\vee_{i2}$ 5
7.	$\neg x_1 \vee x_2$	copying 6
8.	$\bot$	$\neg_e$ 1, 7
9.	$\neg\neg(\neg x_1 \vee x_2)$	$\neg_i$ 1–8
10.	$\neg x_1 \vee x_2$	$\neg\neg_e$ 9
11.	$\neg x_1$	assumption
12.	x_1	assumption
13.	$\bot$	$\neg_e$ 11, 12
14.	x_2	$\bot_e$ 13
15.	$x_1 \to x_2$	$\to_i$ 12–14
16.	x_2	assumption
17.	x_1	assumption
18.	x_2	copying 16
19.	$x_1 \to x_2$	$\to_i$ 17–18
20.	$x_1 \to x_2$	$\vee_e$ 10, 11–15, 16–19

(b) In this question, you are required to fill in the missing proof lines and corresponding annotations (which assumption/proof rule is being applied, and to formulas in which prior lines) to complete the natural deduction proof sketched below.

1.	$p \to q$	premise
2.	$q \to r$	premise
3.	$r \to p$	premise
4.	$\neg p$	premise
5.	$\cdots$	$\cdots$
6.	$\cdots$	$\cdots$
7.	$\cdots$	$\cdots$
8.	$\neg r$	$\neg_i$ 5–7
9.	$\cdots$	$\cdots$
10.	$\cdots$	$\cdots$
11.	$\cdots$	$\cdots$
12.	$\cdots$	$\cdots$
13.	$\cdots$	$\cdots$
14.	$\cdots$	$\cdots$
15.	$\cdots$	$\cdots$
16.	$q \wedge r$	$\vee_e$ 9, 10–12, 13–15
17.	$(p \vee r) \to (q \wedge r)$	$\to_i$ 9–16

2. **Locking Horns!**

Consider the formulas defined below over variables $x_1, \ldots x_{10}$.

$$\begin{aligned}\varphi_1 &= \bigwedge_{i=1}^{8} \big((x_i \wedge x_{i+2}) \rightarrow x_{i+1}\big) \\ \varphi_2 &= \bigwedge_{i=1}^{7} \big((x_i \wedge x_{i+3}) \rightarrow \bot\big) \\ \varphi_3 &= \bigwedge_{i=1}^{2} \top \rightarrow x_{4i}\end{aligned}$$

(a) All of the above formulas are Horn formulas. A student wishes to apply the Horn satisfiability checking algorithm studied in class to find a satisfying assignment or detect unsatisfiability of $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$. Using the algorithm for checking satisfiability of a Horn formula studied in class, find a satisfying assignment of φ or show that it is unsatisfiable.

(b) Now consider a variant of the Horn satisfiability checking procedure, wherein we want to set as many variables to 1 as possible in a satisfying assignment. To achieve this, we must first check for implications for the form $\top \to x_i$. Every such x_i must be set to true. We must also check for implications of the form $x_j \to \bot$ or $x_j \wedge x_k \wedge \cdots \to \bot$. For every such implication, at least one variable on the left hand side of the implication must be set to false. After repeating the above steps and simplifying the resulting implications repeatedly, every remaining implication must be of the form $x_i \to x_j$ or $x_i \wedge x_j \cdots \wedge x_k \to x_l$. All variables appearing in such implications can now be set to 1.

Using the above idea, find a satisfying assignment of the formula given in the question, such that the count of 1s in the satisfying assignment is maximized.

3. **A new proof system (if time permits)**

We've looked at the natural deduction proof system in class. In this question, we will consider a slightly different proof system.

In our new proof system, we have the following proof rules (three axioms that don't rely on any premise, and one inference rule). The way you read an inference rule/axiom remains the same as you did for natural deduction.

$$\frac{}{\varphi \to (\psi \to \varphi)}\ \textsc{Axiom1} \qquad \frac{}{(\varphi \to (\psi \to \rho)) \to ((\varphi \to \psi) \to (\varphi \to \rho))}\ \textsc{Axiom2}$$

$$\frac{}{((\varphi \to \bot) \to \bot) \to \varphi}\ \textsc{Axiom3} \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}\ \textsc{Modus Ponens}\ \textit{or}\ \text{MP}$$

Using only the above rules, show the following:

(a) $\{p\} \vdash q \to p$

In other words, starting from the premise p, derive $q \to p$ using nothing beyond the four rules above. Each line of your proof must be labeled by one of { Axiom1, Axiom2, Axiom3, MP } along with references to previous two lines on which you are applying the Modus Ponens (MP) rule, if you are using the label MP. You will get no marks for using the natural deduction proof rules.

(b) $\{\} \vdash p \to p$

In other words, starting from no premise, derive $p \to p$ using nothing beyond the four rules above. Each line of your proof must be labeled by one of { Axiom1, Axiom2, Axiom3, MP } along with references to previous two lines on which you are applying the Modus Ponens (MP) rule, if you are using the label MP. You will get no marks for using the natural deduction proof rules.