1. [5 + 5 marks] We’ve studied a bit about the theory of abstract interpretation. Consider two abstract domains $\mathcal{A}_i = (A_i, \sqsubseteq_i, \sqcup_i, \sqcap_i, \top_i, \bot_i, \nabla_i)$ for $i \in \{1, 2\}$. Let the corresponding abstraction and concretization functions be $\alpha_i$ and $\gamma_i$ respectively. Assume that $(\alpha_i, \gamma_i)$ form Galois connections for $i \in \{1, 2\}$. In addition, you are told the following facts:

- $\forall a_1 \in A_1, \alpha_1(\gamma_1(a_1)) = a_1$. In other words, $(\alpha_1, \gamma_1)$ forms a Galois insertion.
- $\forall a_2 \in A_2 \exists a_1 \in A_1 \gamma_2(a_2) = \gamma_1(a_1)$. In other words, the shape of concrete state sets that $A_2$ can represent is subsumed by that which $A_1$ can represent.

Show that $(\alpha_2 \circ \gamma_1 : A_1 \rightarrow A_2, \alpha_1 \circ \gamma_2 : A_2 \rightarrow A_1)$ forms a Galois connection between $(A_1, \sqsubseteq_1)$ and $(A_2, \sqsubseteq_2)$.

2. [10 marks] We’ve learnt about threshold widening as a way to improve the precision of the widening operator using a finite set of thresholds. A student proposes the following improvement to threshold widening using a set of thresholds that is not restricted to be finite.

Let $\mathcal{A} = (A, \sqsubseteq, \sqcup, \sqcap, \top, \bot, \nabla)$ be the abstract domain with $\nabla$ being the native widen operator for $\mathcal{A}$. Let $T = (t_0, t_1, t_2 \ldots \top)$ be an increasing chain of elements in $(A, \sqsubseteq)$, i.e. $\bot = t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq \cdots$. The student proposes the following adaptation of $\nabla$, denoted $\nabla_T$:

Given a sequence $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \cdots$, the threshold widened limit (i.e. using $\nabla_T$) is defined as follows.

- Define $z_0 = a_0$ and $z_i = (z_{i-1} \nabla a_i) \cap t_{f(i)}$, where $i \geq 0$ and $f(i)$ is defined by the following relation: $t_{f(i)-1} \subseteq (z_{i-1} \sqcap a_i)$ and $(z_{i-1} \sqcup a_i) \subseteq t_{f(i)}$.
- The threshold widened limit of the sequence $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \cdots$ is given by the native widened limit (i.e. using $\nabla$) of the sequence $z_0 \sqsubseteq z_1 \sqsubseteq \cdots$ obtained above.

Does $\nabla_T$ form a valid widening operator for $\mathcal{A}$? Give reasons for your answer.