Consider a DNN $N$ with $k \geq 3$ layers (layer 1 is the “input” layer, and layer $k$ is the “output” layer). For layer $i$ of the network, let $L_{i-1}$ be the input domain and $L_i$ be the output domain. Thus, the input domain of the overall network $N$ is $L_0$ and its output domain is $L_k$.

Suppose the transformer associated with the slice of the network from layers $i$ through $j$, both inclusive, is $\nu_{i,j}: L_{i-1} \to L_j$. So, the overall transformer of the DNN $N$ is $\nu_{1,k}$.

Let $D_{i-1}: L_{i-1} \times L_{i-1} \to \mathbb{R}^\geq 0$ be a distance metric that maps a pair of inputs of the $i^{th}$ layer to a non-negative real number, such that $D_{i-1}(I, I) = 0$ for every $I \in L_{i-1}$. For $I, J \in L_{i-1}$, we say $D_{i-1}(I, J)$ is the distance between the inputs $I$ and $J$ of layer $i$.

For every layer $i < k$, we also have four parameters $\varepsilon_{1,i}, \varepsilon_{i,k}, \delta_{1,i}, \delta_{i,k} \geq 0$ such that the following two Hoare triples hold:

- $\{ D_0(I_0, J_0) \leq \varepsilon_{1,i} \} I_i \leftarrow \nu_{1,i}(I_0); J_i \leftarrow \nu_{1,i}(J_0); \{ D_i(I_i, J_i) \leq \delta_{1,i} \}$
- $\{ D_i(I_i, J_i) \leq \varepsilon_{i,k} \} I_k \leftarrow \nu_{i+1,k}(I_i); J_k \leftarrow \nu_{i+1,k}(J_i); \{ I_k = J_k \}$.

1. Let $A = \{i \mid \delta_{1,i} \leq \varepsilon_{i,k} \}$. What can you say about the input-output behaviour of the overall network $N$ if:
   (a) [5 marks] $A \neq \emptyset$?
   (b) [5 marks] $A = \emptyset$?

   Explain your answers clearly with reasons.

2. Suppose further that we know that there exist inputs $I^*, J^*$ such that $\nu_{1,k}(I^*) \neq \nu_{1,k}(J^*)$. Indicate with reasons which of the following are necessarily true.
   (a) [5 marks] There is at least one $i < k$ such that $D_i(\nu_{1,i}(I^*), \nu_{1,i}(J^*)) \geq \delta_{1,i}$.
   (b) [5 marks] There is at least one $i < k - 1$ such that $D_i(\nu_{1,i}(I^*), \nu_{1,i}(J^*)) \geq \varepsilon_{i+1,k}$.
   (c) [5 marks] $D_{k-1}(\nu_{1,k-1}(I^*), \nu_{1,k-1}(J^*))$ cannot be arbitrarily large.