CS719 End-semester Exam

Max marks: 75 **Time: 3 hours**

- *The exam is open-book, open-notes and open-material-brought-to-exam-hall.*
- *Be brief, complete and stick to what has been asked. Unnecessarily lengthy solutions may be penalized.*
- *If you need to make any reasonable assumptions, state them clearly. Unreasonable assumptions run the risk of attracting penalty.*
- *If you need to use/cite results covered in class, you may simply cite the result, without going into a formal proof.*
- ***Do not copy from others or indulge in unfair means.***
  Students found indulging in such activities will be awarded the FR grade.

1. (a) *[5 marks]* Show that a lattice $(L; \leq)$ is distributive if and only if $x \vee (y \wedge z) \geq (x \vee y) \wedge z$ for all $x, y, z \in L$.

   (b) Let $(L; \leq)$ be a bounded (i.e. has $\top$ and $\bot$) distributive lattice. In general, some elements of L may have complements. Let $C \subseteq L$ be the set of elements of L that have complements.

   i. *[3 marks]* Is $(C; \leq)$ necessarily a sublattice of $(L; \leq)$?

   ii. *[3 marks]* Is $(L \setminus C; \leq)$ necessarily a sublattice of $(L; \leq)$?

   In each case, you must either give a proof or provide a counter-example.

   (c) Let $(P; \leq)$ be a lattice. Let $(\mathcal{O}(P); \subseteq)$ be the poset of order ideals of P, and $(\mathcal{I}(P); \subseteq)$ be the poset of lattice ideals of P.

   i. *[4 marks]* Give an example of P such that P has both ACC and DCC, but $\mathcal{I}(P)$ has neither ACC nor DCC.

   ii. *[5 marks]* Suppose P is a complete lattice, but there is no finite subset A of P such that $\bigvee A = \top$ or $\bigwedge A = \bot$. Show that $\mathcal{O}(P)$ has neither ACC nor DCC.

2. *[10 marks]* Consider the first order logic sentence:

$$\varphi \equiv (\forall x\, (P(x) \rightarrow \exists y\, Q(x,y))) \wedge (\forall x \forall y\, (Q(x,y) \rightarrow (P(x) \wedge P(y)))) \wedge (\exists x \exists y\, (Q(x,y) \wedge \forall z\ \sim Q(y,z)))$$

   Show using resolution for first order logic that φ is unsatisfiable. Note that φ is not in Skolem Normal form, so you must first Skolemize it, find most general unifiers, and then use resolution to show that the formula is unsatisfiable.

3. Let $(L; \leq)$ be a bounded lattice, i.e. a lattice with top ($\top$) and bottom ($\bot$). For every $u, v \in L$ such that $u < v$, we define the *closed interval* $[u, v]$ to be the poset $(\{x \mid u \leq x \leq v\}; \leq)$.

(a) *[5 marks]* Is $[u, v]$ a lattice for every $u, v \in L$ such that $u < v$? Either give a proof or provide a counterexample.

(b) *[10 marks]* For every closed interval $[u, v]$ in L and for every $a \in [u, v]$, a *relative complement* of a with respect to $[u, v]$ is an element $b \in [u, v]$ such that $a \wedge b = u$ and $a \vee b = v$. The complement (as studied in class) is easily seen to be a special case of this, i.e. relative complement with respect to $[\bot, \top]$.

Show that if L is modular and $a \in L$ has a complement, then a also has a relative complement $\widehat{a}$ with respect to every closed interval $[u, v]$ in L such that $u \leq a \leq v$.

(c) *[5 marks]* A closed interval $[u, v]$ in L is said to be complemented if every $a \in [u, v]$ has a relative complement with respect to $[u, v]$. The lattice L is said to be relatively complemented if every closed interval $[u, v]$ in L is complemented.

Is it possible for L to be complemented (i.e. every element has a complement), but not be relatively complemented? Either give a proof or provide a counter-example.

Hint: The previous question asked you to show that this is not possible if L is modular.

(d) *[10 marks]* Suppose we are told that for every closed interval $[u, v]$ in L and $a \in [u, v]$, either a has a unique relative complement with respect to $[u, v]$ or no relative complement with respect to $[u, v]$. In other words, relative complements, when they exist, are unique. Show that L must be distributive.

4. After spending a sleepless night on the eve of CS719 end-sem exam trying to understand how first order logic can (and cannot) be used to describe properties of directed graphs, two students X and Y have come up with two different conclusions. In the following, all first order logic sentences are assumed to be on the vocabulary $\Sigma = \{E, =\}$, where E is a binary predicate.

- X thinks that if a certain property of graphs cannot be expressed by any FOL sentence φ on Σ, then such a property cannot be expressed even by an infinite family Γ of FOL sentences on Σ. Although the student isn't sure of the exact argument, she thinks this should be provable using the Compactness Theorem of first order logic.
- Y thinks that every property of graphs can be expressed by an infinite family Γ of FOL sentences on Σ. Although he doesn't have a proof yet, he intuitively feels that since there are no bounds on the cardinality of Γ, it should be possible to express every property of graphs using a sufficiently large number (possibly infinite) of suitably designed FOL sentences on Σ.

Note that when we say that a FOL sentence φ (or a family of FOL sentences Γ) expresses a property $\mathcal{P}$ of directed graphs, we mean the following. For every directed graph G with property $\mathcal{P}$, the corresponding Σ-structure M_G is a model of φ (or a model of every FOL sentence in Γ). Similarly, every model M_G of φ (or of Γ) represents a graph G with property $\mathcal{P}$.

Show that both X and Y are wrong. In other words:

(a) *[7.5 marks]* Show that there are properties of graphs that cannot be expressed by an individual FOL sentence on Σ, but can be expressed by an infinite family of FOL sentences on Σ.

(b) *[7.5 marks]* Show that there are properties of graphs that cannot be expressed by any infinite family of FOL sentences on Σ.