CS719 Graded Homework #2

Max Marks: 50 **Due date: Oct 26, 2009 (5 pm)**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Discussion among students is fine, but the solution you turn in must be your own solution in your own words. Cases of copying or indulgence in unfair means will be severely penalized, including award of FR grade.***

Problems on lattices and partial orders:

1. *[Adapted from Problem 2.28, Davey and Priestley]* Let $(P, \leq)$ be an ordered set. A possibly empty subset K of P is said to be *order-convex* if the following holds:

$$\forall a, b, x\ ((a \in K) \wedge (b \in K) \wedge (x \in P) \wedge (a \leq x \leq b) \Rightarrow x \in K)$$

Let $\mathcal{K}(P) = \{K \subseteq P \mid K \text{ is an order-convex subset of } P\}$.

   (a) *[2 marks]* List all subsets of the poset $\mathbf{2^2}$ that are order-convex.

   (b) *[3 marks]* Show that for every $n \geq 2$, there exists a partial order $(P, \leq)$ such that (i) $|P| = n$, (ii) $(P, \leq)$ is not an antichain, and (iii) $\mathcal{K}(P) = \wp(P)$. You must specify finite posets either by means of Hasse diagrams, or by listing the covering relation of the poset.

   (c) *[5 marks]* Prove that for any poset $(P, \leq)$, $\mathcal{K}(P)$ is a topped intersection structure on P. In other words, $(\mathcal{K}(P), \subseteq)$ is closed under set intersection and has a top element.

2. *[10 marks]* Let $(P, \leq)$ be a complete lattice. Consider the map $\varphi : P \rightarrow \wp(P)$ defined as $\varphi(a) = \uparrow a$ for all $a \in P$, i.e. $\varphi(a)$ is the principal order filter defined by a. Let $\varphi(P) \subseteq \wp(P)$ be the set of all principal order filters of P. Show that $(\varphi(P), \subseteq)$ is a complete lattice.

3. *[Adapted from problem 2.25, Davey and Priestley]* A subset A of $\aleph$ is said to be *co-finite* if $\aleph \setminus A$ is finite.

   (a) *[4 marks]* Let $\mathcal{L}_1 = \{S \in \wp(\aleph) \mid S \text{ is co-finite }\}$. Show that $(\mathcal{L}_1, \subseteq)$ is a lattice with join and meet defined by set union and set intersection, respectively.

   (b) *[4 marks]* Let $\mathcal{L}_2 = \{S \in \wp(\aleph) \mid S \text{ is co-finite or } S \text{ is finite }\}$. Show that $(\mathcal{L}_2, \subseteq)$ is a lattice with join and meet defined by set union and set intersection, respectively.

   (c) Let $A_n = \aleph \setminus \{2, 4, \ldots 2n\}$, i.e., A_n is the set of all natural numbers except the first n even numbers. Let B_m be a set such that $|B_m| = m$ and $B_m \subseteq A_n$ for all $n \geq 1$.

      i. *[2 marks]* Show that there exists a set $B_m \subseteq \aleph$ satisfying the above requirements for all $m \geq 1$.

      ii. *[3 marks]* Show that $B = \bigcup_{m \geq 1} B_m$ is not co-finite.

      iii. *[2 mark]* From the results of the previous two subquestions, show that neither $(\mathcal{L}_1, \subseteq)$ nor $(\mathcal{L}_2, \subseteq)$ are complete lattices.

4. *[5 marks]* Let $(L; \wedge, \vee)$ be a lattice. Define $\mathcal{I}(L) = \{S \mid S \text{ is a sublattice of } L, \text{ and } \forall a \in S, b \in L,\ (a \wedge b) \in S\}$. Show that $(\mathcal{I}(L); \subseteq)$ is a complete lattice.

5. Let $(L; \wedge, \vee)$ be a lattice. An ideal I of L is called *proper* if $I \neq L$. A proper ideal is said to be *prime* if for every $a, b \in L$, $a \wedge b \in I$ implies either $a \in I$ or $b \in I$. Let $Spec(L)$ denote the set of prime ideals of L, and let $f : L \to M$ be a lattice homomorphism.

   (a) *[5 marks]* Show that if P is an ideal of M, then $f^{-1}(P) = \{x \mid x \in L, f(x) \in P\}$ is an ideal of L.

   (b) *[5 marks]* Suppose we know that $P \in Spec(M)$. Is $f^{-1}(P)$ necessarily in $Spec(L)$? Give a counterexample or prove that $f^{-1}(P) \in Spec(L)$.