CS719 Graded Homework #3

Max Marks: 75 **Due date: Nov 5, 2009 (5 pm)**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Discussion among students is fine, but the solution you turn in must be your own solution in your own words. Cases of copying or indulgence in unfair means will be severely penalized, including award of FR grade.***

Problems on lattices:

1. (a) *[5 marks]* Let L and M be lattices, where L has a bottom element 0, and let $f : M \to L$ be a lattice homomorphism. The *ideal kernel* of f, denoted $ker(f)$, is defined to be $f^{-1}(0) = \{x \mid x \in M, f(x) = 0\}$. Show that $ker(f)$ is an ideal of M (hence the name "ideal kernel").

   (b) *[5 marks]* Let $\mathbf{M_3} = (\{\bot, \top, a, b, c\}; \leq)$ be the lattice with covering relation: $\bot -\!\!< a, \bot -\!\!< b, \bot -\!\!< c, a -\!\!< \top, b -\!\!< \top, c -\!\!< \top$. Clearly, $\{\bot, a\}$ is an ideal of $\mathbf{M_3}$. Show that $\{\bot, a\}$ cannot be $ker(f)$ for any lattice homomorphism $f : \mathbf{M_3} \to L$, where L is a lattice with a bottom element.

2. *[10 marks]* An ideal I of a lattice $(L; \leq)$ is said to be *prime* if $I \subset L$ and $\forall a, b \in L,\ a \wedge b \in I \Rightarrow a \in I$ or $b \in I$. Show that every ideal of a lattice is prime if and only if L is a chain.

3. *[10 marks]* Show that a nonempty subset S of lattice L is a convex sublattice of L if and only if $S = I \cap F$, where I is an ideal of L and F is a filter of L.

   [Recall from the previous homework that a convex subset S of a poset $(P; \leq)$ is one such that $\forall x, y \in S, \forall z \in P,\ x \leq z \leq y \Rightarrow x \in S$.]

4. We have seen in class that if a lattice L has ACC and DCC, every element $a \in L$ can be expressed as the join of a finite subset of $\mathcal{J}(L)$, and also as the meet of a finite subset of $\mathcal{M}(L)$. For every element $a \in L$, define $J(a)$ to be the cardinality of the smallest subset of $\mathcal{J}(L)$ whose join gives a. Similarly, we define $M(a)$ to be the cardinality of the smallest subset of $\mathcal{M}(L)$ whose meet gives a. Suppose we are told that $\max_{a \in L} J(a) = m$ and $\max_{a \in L} M(a) = n$, where $m, n \in \aleph$.

   (a) *[5 marks]* Show that there are at least $\max(m, n)$ chains in L, none of which is a subchain of the other.

   (b) *[5 marks]* Show that if $\max(m, n) > 1$ there is at least one (not necessarily maximal) chain in L of length $\geq \max(\lfloor \log_2 m \rfloor, \lfloor \log_2 n \rfloor)$.

   (c) *[5 marks]* Show that every $a \in L$ can be written as a finite meet-of-joins of $\mathcal{J}(L)$, and also as a finite join-of-meets of $\mathcal{M}(L)$. In other words, show that $a = \bigwedge_{i=1}^{p} (\bigvee_{j=1}^{k_i} a_{i,j})$, where p, $k_1, \ldots k_p \in \aleph$ and $a_{i,j} \in \mathcal{J}(L)$. Similarly, show that $a = \bigvee_{i=1}^{q} (\bigwedge_{j=1}^{l_i} b_{i,j})$, where q, $l_1, \ldots l_q \in \aleph$ and $b_{i,j} \in \mathcal{M}(L)$.

5. Let (α, γ) be a Galois connection on the pair of lattices $((P; \leq_P), (Q; \leq_Q))$. Let $\alpha(P) = \{x \in Q \mid \exists y \in P.\, \alpha(y) = x\}$, and $\gamma(Q) = \{y \in P \mid \exists x \in Q.\, \gamma(x) = y\}$.

   (a) *[10 marks]* Show that $\forall y \in alpha(P),\ \alpha(\gamma(y)) = y$ and $\forall x \in \gamma(Q),\ \gamma(\alpha(x)) = x$.

   (b) *[5 + 5 marks]* Show that de Morgan's laws hold in $\alpha(P)$ and $\gamma(Q)$. In other words, show that:

i. For every $a, b \in \alpha(P)$, $\gamma(a \wedge_Q b) = \gamma(a) \vee_P \gamma(b)$ and $\gamma(a \vee_Q b) = \gamma(a) \wedge_P \gamma(b)$.

ii. For every $u, v \in \gamma(Q)$, $\alpha(u \wedge_P v) = \alpha(u) \vee_Q \alpha(v)$ and $\alpha(u \vee_P b) = \gamma(a) \wedge_Q \gamma(b)$.

6. *[10 marks]* Show that a pair of maps (α, γ) between posets $(P; \leq_P)$ and $(Q; \leq_Q)$ is a Galois connection if and only if the following condition is satisfied:

$$\forall p \in P,\ \forall q \in Q,\ \alpha(p) \geq_Q q \Leftrightarrow p \leq_P \gamma(q)$$

The above condition is also sometimes taken as an alternative definition of a Galois connection.