CS719 Solutions to Graded Homework #4

- *The solutions have been intentionally kept lengthy to help clarify doubts.*
- *Your solutions need not be this lengthy, and results covered in class can be used by simply mentioning them.*

1. Let L be a Boolean lattice. Suppose L has an infinite ascending chain: $a_0 < a_1 < a_2 < \dots$. We know that for a Boolean lattice, if $a_i < a_j$, then $a_i' > a_j'$ (a quick proof: $(a_i \vee a_j)' = a_j'$ and also by de Morgan's law, $(a_i \vee a_j)' = a_i' \wedge a_j'$; hence $a_j' \leq a_i'$, but $a_i < a_j \Rightarrow a_i' \neq a_j'$ since complements are unique in a Boolean lattice). Therefore, we get an infinite descending chain $a_0' > a_1' > a_2' > \dots$. Thus, if L doesn't have ACC, then it doesn't have DCC as well. The argument above is completely symmetric with respect to $<$ and $>$. Therefore, by a similar argument, if L doesn't have DCC, it doesn't have ACC either. It follows that L has DCC iff it has ACC.

2. (a) Let L be a distributive lattice, and suppose that $((a \vee b) = (c \vee b))$ and $((a \wedge b) = (c \wedge b))$ for some $a, b, c \in L$. We have $a = a \vee (a \wedge b) = a \vee (c \wedge b) = (a \vee c) \wedge (a \vee b) = (a \vee c) \wedge (c \vee b) = (a \wedge b) \vee c = (c \wedge b) \vee c = c$.

   (b) There seem to be multiple ways to approach this problem. I'm taking the route suggested by the hint given in the book. This route involves showing that if all chains in L have length $\leq k$ and L is distributive, then $\mathcal{J}(L)$ is finite. Furthermore, if all chains in L have length $\leq k$, L clearly has DCC. Since in a lattice with DCC, every element can be expressed as the join of a finite subset of $\mathcal{J}(L)$. it then follows that there are only finitely many elements in L. The crucial part of the proof is therefore to show that $\mathcal{J}(L)$ is finite.

   Since all chains in L have length $\leq k$, there are two consequences:

   - There exists a maximal chain, i.e. a chain with maximum length. Let C be such a maximal chain.
   - Every maximal chain has the same smallest element and the same largest element (otherwise you could take the distinct smallest elements of two maximal chains, compute their meet and obtain a longer chain; same argument holds for the largest elements). This means that the lattice has a bottom ($\perp$) and a top ($\top$), and these are part of every maximal chain.

   Let $\mathcal{J}(L)$ be the set of join-irreducible elements of L. We will first show that $\mathcal{J}(L)$ is non-empty. Let C be a maximal chain. As argued above, $\perp$ is in C. Since C is of finite length, there exists an element b in C that is covered by $\perp$, i.e. $\perp < b$ and there doesn't exist any other element $d \in C$ such that $\perp < d < b$. We claim that b is join-irreducible. Why? Assume the contrary. Then, there exist $x, y \in L$ such that $b = x \vee y$ and $\perp < x, y < b$. Therefore, $C \cup \{x\}$ is a chain containing one more element than C. This is impoosible since C is a maximal chain. Therefore, b is join-irreducible, and hence $\mathcal{J}(L)$ is non-empty.

   Consider the mapping $\varphi : \mathcal{J}(L) \to C$ defined by $\varphi(x) = \bigwedge(\uparrow x \cap C)$. Thus, $\varphi(x)$ is the smallest element in C that is $\geq x$. Clearly, since $\top$ is a part of every maximal chain, $\varphi(x)$ is well-defined for every $x \in \mathcal{J}(L)$.

We now claim that φ is a one-to-one mapping. We prove this by contradiction.

Suppose there exist $x, y \in \mathcal{J}(L)$ such that $x \neq y$ and $\varphi(x) = \varphi(y) = a$. Since x and y are join-irreducible, $x > \bot$ and $y > \bot$ (recall $\bot$ is not join-irreducible). Since $\varphi(x) \geq x > \bot$ and $\varphi(y) \geq y > \bot$, it follows that $\varphi(x) = \varphi(y) = a > \bot$. Since $\bot \in C$ and C is finite, there exists a cover of a in C. Let this cover be e. In other words, there exists $e \in C$ such that $e < a$ and there is no $f \in C$ such that $e < f < a$. Since C is also a maximal length chain, we can say further. Specifically, there is no f in L such that $e < f < a$. Otherwise, we could have obtained the chain $C \cup \{f\}$, that is longer than the maximal length chain C.

Since $a = \bigwedge(\uparrow x \cap C) = \bigwedge(\uparrow y \cap C)$ and C is a chain, it is easy to see that $x \leq a$ and $y \leq a$. We can, in fact, say more, i.e. $x < a$ and $y < a$ (strict inequalities). Why? Suppose the contrary and without loss of generality, let $x = a$, if possible. Since $y \leq a$ and $e < a$, therefore $(y \vee e) \leq a$. If $e < (y \vee e) < a$, we would have an element f $(= y \vee e)$ such that $e < f < a$. However, we have seen above that such an f doesn't exist. Therefore, we must have either $e = (y \vee e)$ or $(y \vee e) = a$. If $e = (y \vee e)$, then $y \leq e$, and since $e \in C$, we should have $\varphi(y) = \bigwedge(\uparrow y \cap C) \leq e$. However, we know that $\varphi(y) = a > e$. If $(y \vee e) = a$, then since $x(= a)$ is join-irreducible and $e < a$, we must have $y = a$. However, this contradicts the fact that $x \neq y$. Therefore, we must have $x < a$ and similarly, $y < a$.

Since $e < a$ and $e \in C$, it follows that $x \not\leq e$ and $y \not\leq e$ (otherwise $\varphi(x)$ or $\varphi(y)$ would be $\leq e$, while we know $\varphi(x) = \varphi(y) = a > e$). So what is the relation of e with x and y? We have the following two possibilities:

- $e < x(< a)$ or $e < y(< a)$: None of these cases are possible since we know that there is no $f \in L$ such that $e < f < a$.
- $e || x$ and $e || y$: This is the only remaining possibility.

Now, consider the element $x \wedge (e \vee y)$. Since $e, y < a$, therefore $(e \vee y) \leq a$. Suppose $(e \vee y) < a$. Since $e || y$ as seen above, therefore we have $e < (e \vee y)$. But, this gives us an element $f(= e \vee y)$ such that $e < f < a$. We know that this is not possible. Therefore, we must have $(e \vee y) = a$. But since $x < a$, it follows that $x \wedge (e \vee y) = x \wedge a = x$.

Since L is distributive, the above equality can be re-written as $(x \wedge e) \vee (x \wedge y) = x$. We have seen above that $e || x$. Therefore, $(x \wedge e) < x$. Since x is join-irreducible and $(x \wedge e) < x$, it follows that $(x \wedge y) = x$. However, we also know that $x \neq y$. Therefore, we must have $x < y$.

Note that the above argument can be repeated by swapping the roles of x and y. Thus, we could start from the element $y \wedge (e \vee x)$ and show that $y < x$. This contradicts the previous conclusion $x < y$.

Therefore, our initial assumption must be wrong. In other words, if $x, y \in \mathcal{J}(L)$ and $x \neq y$, then $\varphi(x) \neq \varphi(y)$. However, the range of φ is C, a finite chain. Since φ is one-to-one (injective), it follows that its domain is also finite. In other words, $\mathcal{J}(L)$ is finite.

3. (a) Let L be a finite distributive lattice. Since every finite lattice has a top $\top$ and a bottom $\bot$, so does L. Therefore, for every $a \in L$, the set $S_a = \{b \mid b \in L, b \wedge a = \bot\}$ always includes $\bot$, and is hence non-empty.

   Suppose $b_1, b_2 \in S_a$. Therefore, $b_1 \wedge a = \bot = b_2 \wedge a$. Since, L is distributive, $(b_1 \vee b_2) \wedge a = (b_1 \wedge a) \vee (b_2 \wedge a) = \bot \vee \bot = \bot$. Therefore, $(b_1 \vee b_2) \in S_a$. In other words, S_a is closed under $\vee$. Similarly, if $b_1 \in S_a$ and $c \in L$ is such that $c \leq b_1$, then $c \wedge a \leq b_1 \wedge a = \bot$. It follows that $c \wedge a = \bot$, and hence $c \in S_a$. Thus, S_a is a lattice ideal of L. Since $S_a \subseteq L$ and L is finite, therefore S_a is finite as well. Every finite ideal of a lattice has a maximum element (why? because it is closed under $\vee$). Hence, S_a has a unique maximum element.

(b) Let L be a Boolean lattice, and $a \in L$. The set S_a as defined above is non-empty since $a' \wedge a = \bot$. Now, let $b \in S_a$. Therefore, $b \wedge a = \bot$.

Consider $b \wedge (a \vee a') = b \wedge \top = b$. Since L is ditributive (every Boolean lattice is distributive), therefore b can also be written as $(b \wedge a) \vee (b \wedge a') = \bot \vee (b \wedge a') = b \wedge a'$. Therefore, $b = b \wedge a'$. In other words, $b \leq a'$. Therefore, a' is the unique maximum element in S_a.

(c) Consider the lattices $L_1 = \aleph \times \aleph$ and $L_2 = \Re_{\geq 0} \times \Re_{\geq 0}$, where $\Re_{\geq 0}$ is the set of all non-negative real numbers.

- L_1 is countable, but L_2 is not. Therefore, they cannot be isomorphic.
- Since every chain is a distributive lattice, and products of distributive lattices are distributive lattices, therefore both L_1 and L_2 are distributive lattices.
- $(1, 1)$ serves as the bottom for $L_1 = \aleph \times \aleph$ Consider the element $a = (1, 2) \in L_1$. Here, $S_a = \{(n, 1) \mid n \in \aleph\}$. Clearly, S_a doesn't have a maximum element.
  Similarly, $(0, 0)$ serves as the bottom for L_2. By similar reasoning as above, if $a = (0, 1) \in L_2$, $S_a = \{(r, 0) \mid r \in \Re_{\geq 0}\}$ has no maximum element.

4. There could be multiple solutions to this question. I am providing one of the simplest examples.

(a) Consider the sentence $\varphi \equiv \varphi_1 \wedge \varphi_2$ (not exactly satisfying the conditions of the question, but we'll fix this soon), where $\varphi_1 \equiv (\exists x \forall y\, \neg(f(y) = x))$ and $\varphi_2 \equiv (\forall x \forall y\, \neg(x = y) \rightarrow \neg(f(x) = f(y)))$. φ_1 asserts that f is not a surjective function, and φ_2 asserts that f is injective.

We claim that every model of φ must have infinitely many elements in the universe. Why? Suppose the contrary, and let (U^M, f^M) be a model of φ, where $|U^M| = k \in \aleph$. From φ_2, it follows that $f^M : U^M \rightarrow U^M$ is an injective function. From φ_1, it follows that f^M is not surjective. Therefore, the domain of f^M has the same cardinality as that of a strict subset of the range. *Since the domain and range of f^M are finite, this implies that the cardinality of the range is higher than that of the domain.* However, both the domain and range of f^M are U^M. Therefore, $k > k$ for some $k \in \aleph$, which is an absurdity.

Note that the argument italicized above does not apply if the domain and range of f^M are infinite. Indeed $g : \aleph \rightarrow \aleph$ defined by $g(n) = 2.n$ is injective but not surjective, but the set of even natural numbers has the same cardinality as the set of all natural numbers. It is possible to define an injective function from the set of even natural numbers to the set of all natural numbers and vice versa, and therefore by the Bernstein-Schröder theorem, there is a bijection between these two sets. In other words, their cardinalities are the same.

So now we have a sentence φ such that all models of φ are necessarily infinite. But this φ doesn't suit our purpose, since it contains a function symbol f. We will now obtain a sentence ψ without f, but with uninterpretted predicates such that for every model M of φ, there is a model M' of ψ with the same universe, and vice versa. Note that φ and ψ are not semantically equivalent – they cannot be, since their signatures are not the same and hence their models cannot be the same. However, φ and ψ are equisatisfiable. Even more importantly, for every model M of φ, there is a model M' of ψ with the same universe, and vice versa. Since we have shown above that every model of φ must have an infinite universe, every model of ψ must also have an infinite universe.

The only part remaining is to obtain ψ from φ. But this is a standard trick: replacing functions in a FOL formula with appropriate predicates. After all, an n-ary function is a special case of an $n+1$-ary predicate. Since we have a unary function f, intuitively, we want to have an uninterpretted binary predicate $P_f(z, w)$ that evaluates to True exactly for one value of w for every value of z. This would then encode a functional relation between w and z, similar to what $w = f(z)$ captures. This is done through ψ_1 below. Once we have such a predicate P_f, we must replace every occurence of $f(z)$ with w and add the constraint that $P_f(z, w)$ evaluates to True. Thus, we have $\psi \equiv \psi_1 \wedge \psi_2 \wedge \psi_3$, where

- $\psi_1 \equiv (\forall z \exists w\, P_f(z, w)) \wedge (\forall z \forall w_1 \forall w_2\, P_f(z, w_1) \wedge P_f(z, w_2) \rightarrow (w_1 = w_2))$. This asserts that there is a functional dependence of w on z if $P_f(z, w) =$ True.
- $\psi_2 \equiv (\forall x \exists y\, P_f(x, y))$. This, along with ψ_1, captures the constraint specified by φ_1.
- $\psi_3 \equiv (\forall x \forall y \forall w_1 \forall w_2\, \neg(x = y) \wedge P_f(x, w_1) \wedge P_f(y, w_2) \rightarrow \neg(w_1 = w_2))$. This, along with ψ_1, captures the constraint specified by φ_2.

Note that ψ uses no function symbols, but an uninterpretted predicate P_f. How do we show that for every model of φ, there is a model of ψ with the same universe and vice versa? Let $M = (U^M, f^M)$ be a model of φ. It follows that $N = (U^M, P_f^N)$ is a model of ψ_1, ψ_2 and ψ_3, where $P_f^N(x, y) =$ True iff $y = f^M(x)$. Similarly, let $N = (U^N, P_f^N)$ be a model of ψ. Then $M = (U^N, f^M)$ is a model of φ_1 and φ_2, where $y = f(x)$ iff $P_f^N(x, y) =$ True.

(b) Not every model of the sentence ψ above has a finite universe.

Let $M = (\aleph, P_f^M)$, where $P_f^M(x, y)$ is defined to be True for all $x, y \in \aleph$. Clearly, M doesn't satisfy ψ_1, and therefore doesn't satisfy ψ. In other words, $M \models \neg\psi$ although the universe of M is infinite.

5. Given $\phi \equiv \forall x\, \exists y\, \forall z\, (P(x, y) \vee \neg P(z, y))$ and $\psi \equiv \forall x\, ((\exists y\, P(x, y)) \vee (\forall y\, \neg P(y, x)))$.

Let $M = (U^M, P^M) \models \psi$. Let $U_1^M = \{x \mid \exists y\, P^M(x, y) = \text{True}\}$ and $U_2^M = U^M \setminus U_1^M$.

Since $M \models \psi$, we have the following:

- $((\exists y\, P^M(x, y)) \vee (\forall y\, \neg P^M(y, x))) =$ True for all $x \in U_1^M$. This, of course, follows trivially from the definition of U_1^M.
- $((\exists y\, P^M(x, y)) \vee (\forall y\, \neg P^M(y, x))) =$ True for all $x \in U_2^M$. From the definition of U_2^M, this implies that $\forall y\, \neg P^M(y, x) =$ True for all $x \in U_2^M$.

When evaluating ϕ on M, we must try to select a value of y for every value of x (similar in spirit to a Skolem function) such that for all values of z, $\left(P^M(x, y) \vee \neg P^M(z, y)\right)$ evaluates to True. We choose to select y in the following manner.

- If $x \in U_1^M$, then by definition, $\exists y\, P^M(x, y) =$ True. Let y_x be the corresponding value of y, i.e., $P^M(x, y_x) =$ True. We therefore choose $y = y_x$ for every $x \in U_1^M$ when evaluating ϕ on M. Since $P^M(x, y_x) =$ True, it follows that $\forall z\, \left(P^M(x, y_x) \vee \neg P^M(z, y)\right) =$ True.
- If $x \in U_2^M$, then as seen above $\forall y\, \neg P^M(y, x) =$ True. We therefore choose $y = x$ for every $x \in U_2^M$ when evaluating ϕ on M. It then follows that $(\forall z\, \neg P^M(z, y)) =$ True and therefore, $\forall z\, \left(P^M(x, y) \vee \neg P^M(z, y)\right) =$ True.

Thus, with the above choice of y, we have $\exists y \forall z \left(P^M(x,y) \vee \neg P^M(z,y)\right) = \mathsf{True}$ for every $x \in U^M$. Hence $M \models \phi$. In other words, $\psi \rightarrow \phi$.

6. (a) Consider the infinite family of sentences $\Gamma = \{\psi_i \mid i \in \aleph\}$, where $\psi_i = \exists x_1 \ldots \exists x_i \bigwedge_{1 \leq j < k \leq i} \neg(x_j = x_k)$. Thus, a model of ψ_i must have *at least* i elements, for every $i \in \aleph$. It follows that a model of Γ must have infinitely many elements.

   Now consider $\Gamma' = \Gamma \cup \{\phi\}$. Let S be a finite non-empty subset of Γ'. If $S \cap \Gamma = \emptyset$, then $S = \{\phi\}$. Therefore, S can be satisfied by a model which has exactly two elements in its universe. If $S \cap \Gamma \neq \emptyset$, let k be the highest subscript of a sentence from Γ that appears in S. Consider a model M with a universe having k' elements, where k' is the smallest even natural number $\geq k$. Clearly, $M \models \psi_i$ for every $\psi_i \in S \cap \Gamma$ and $M \models \phi$ as well. Therefore, $M \models S$. Thus, every finite subset of Γ' is satisfiable, and hence, by the Compactness Theorem, Γ' has a model. Since $\Gamma \subset \Gamma'$, every model of Γ' is also a model of Γ. However, as seen above, every model of Γ must have an infinite universe. Therefore, every model of Γ' necessarily has an infinite universe. Since Γ' is satisfiable, we also know from the Lowenheim-Skolem Theorem that there exists a countable model of Γ'. Thus, there exists a countably infinite model M_1 of Γ'. Since $\phi \in \Gamma'$, therefore, $M_1 \models \phi$.

   Let $\Gamma'' = \Gamma \cup \{\neg\phi\}$. By repeating the same argument as above, and choosing models of odd but finite sizes, we can show that every finite subset of Γ'' is satisfiable. Therefore, by the Compactness Theorem, Γ'' is satisfiable. Since $\Gamma \subset \Gamma''$, every model of Γ'' is a model of Γ, and therefore has infinitely many elements. By the Lowenheim-Skolem Theorem, Γ'' has a countable model. Therefore, Γ'' has a countably infinite model. Let this be M_2. Since $\neg\phi \in \Gamma''$, $M_2 \models \neg\phi$.

   (b) Since the signature of Γ' as well as Γ'' is $\{=\}$, and equality is an interpetted predicate, therefore both M_1 and M_2 simply have a universe each (no interpretations of predicates or functions). In other words, M_1 is simply a countably infinite set and so is M_2.

   We now claim that M_1 and M_2 are isomorphic. Since both M_1 and M_2 are countably infinite, there exist bijections $f_1 : M_1 \rightarrow \aleph$ and $f_2 : M_2 \rightarrow \aleph$. Therefore, $f_2^{-1} \circ f_1 : M_1 \rightarrow M_2$ defined by $f_2^{-1} \circ f_1(x) = f_2^{-1}(f_1(x))$ is also a bijection. Let us denote $f_2^{-1} \circ f_1$ by f.

   The only predicate of concern is $=$. Since f is a bijection, it is easy to see that for every $x_1, y_1 \in M_1$, $(x_1 = y_1)$ iff $(f(x_1) = f(y_1))$. Similarly, since f^{-1} is a bijection, it follows that for every $x_2, y_2 \in M_2$, $(x_2 = y_2)$ iff $(f^{-1}(x_2) = f^{-1}(y_2))$. Therefore, M_1 and M_2 are isomorphic. Hence for every formula ψ on the signature $\{=\}$, $M_1 \models \psi$ iff $M_2 \models \psi$ (given as part of the question). We already know that $M_2 \models \neg\phi$. Therefore, $M_1 \models \neg\phi$ as well. However, we have already seen that $M_1 \models \phi$. This gives a contradiction. Hence our original assumption must be wrong. In other words, there cannot be a FOL formula ϕ on the signature $\{=\}$ such that $M \models \phi$ iff M has an even number of elements in its universe.