CS719 Graded Homework #4

Max Marks: 55 **Due date: Nov 13, 2009 (5 pm)**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Discussion among students is fine, but the solution you turn in must be your own solution in your own words. Cases of copying or indulgence in unfair means will be severely penalized, including award of FR grade.***

Problems on distributive and boolean lattices:

1. *[5 marks]* Prove that a Boolean lattice has DCC if and only if it has ACC.

2. *[5 + 5 marks, adapted from Problems 4.9 and 4.19 of Davey & Priestley]* Let L be a distributive lattice. Prove the following:

   (a) For all $a, b, c \in L$, $((a \vee b) = (c \vee b)$ and $(a \wedge b) = (c \wedge b)) \Rightarrow a = c$

   (b) If there exists $k \in \aleph$ such that every chain in L is of length $\leq k$, then L must be finite.

3. *[5 + 5 + 5 marks, adapted from endsem exam of Spring 2009]* A lattice L with bottom element $\bot$ is said to be *pseudo-complemented* if for each $a \in L$, there exists a unique $a^* \in L$ such that (i) $a^* \wedge a = \bot$, and (ii) $b \leq a^*$ for every $b \in L$ that satisfies $b \wedge a = \bot$. In other words, the set $\{b \mid b \in L, b \wedge a = \bot\}$ has a unique maximum element for every $a \in L$.

   (a) Show that every finite distributive lattice is pseudo-complemented.

   (b) Show that every Boolean lattice (even if infinite) is pseudo-complemented.

   (c) Give two non-isomorphic distributive lattices that are not pseudo-complemented. Simply mentioning the lattices will fetch no marks. Your answer must explain why they are non-isomorphic, distributive and not pseudo-complemented.

Problems on first-order logic and structures:

4. *[5 + 5 marks]*

   (a) Give an example of a first-order logic sentence without function symbols (but possibly with equality) such that the sentence is satisfiable, and every model of the sentence necessarily has an infinite universe. You must use no interpretted predicate other than equality in your sentence. You must also explain why every model of your sentence is necessarily infinite.

   (b) Let the sentence in your solution to the above question be ϕ. Clearly, $\neg\phi$ is satisfiable, since any model with a finite universe doesn't satisfy ϕ and hence satisfies $\neg\phi$. Does every model of $\neg\phi$ necessarily have a finite universe? You must either show that there is an infinite model of $\neg\phi$ or prove that every model of $\neg\phi$ has a finite universe.

5. *[5 marks]* Consider the following first-order logic sentences:
   $\phi \equiv \forall x\, \exists y\, \forall z\ (P(x, y) \vee \neg P(z, y))$, and
   $\psi \equiv \forall x\, ((\exists y\, P(x, y)) \vee (\forall y\, \neg P(y, x)))$.

Prove that $\psi \to \phi$ by demonstrating that every model $M = (U^M, P^M)$ that satisfies ψ also satisfies ϕ. In other words, we wish to reason about the models of ϕ and ψ in order to prove the semantic entailment of the two sentences.

[Hint: Partition the universe U^M of a model M into two parts, one containing all elements x that satisfy $\exists y\, P(x, y)$, and the other containing all other elements of U^M. Of course, you are welcome to use a different reasoning as well.]

6. *[5 + 5 marks]* Let Σ be a relational signature (i.e., no function symbols). Two Σ-structures M_1 and M_2 are said to be isomorphic iff there exists a bijection $f : U^{M_1} \to U^{M_2}$ such that for every k-ary predicate P in Σ and every k-tuple $(a_1, a_2, \ldots a_k) \in (U^{M_1})^k$ and $(b_1, b_2, \ldots b_k) \in (U^{M_2})^k$, we have the following: $P^{M_1}(a_1, a_2, \ldots a_k) = P^{M_2}(f(a_1), f(a_2), \ldots f(a_k))$ and $P^{M_2}(b_1, b_2, \ldots b_k) = P^{M_1}(f^{-1}(b_1), f^{-1}(b_2), \ldots f^{-1}(b_k))$. It can be shown (try this as an ungraded exercise) that if M_1 and M_2 are isomorphic Σ-structures, then for every first-order logic sentence ϕ on the signature Σ, $M_1 \models \phi$ iff $M_2 \models \phi$.

   Now consider $\Sigma = \{=\}$, i.e., the signature containing only the equality predicate. We wish to show that there does not exist any first-order logic sentence ϕ over Σ such that $M \models \phi$ iff M has an even number of elements in its universe.

   Suppose, by way of contradiction, that there indeed existed such a formula ϕ.

   (a) Show using the Compactness Theorem and Lowenheim-Skolem Theorem that we must then have countably infinite Σ-structures M_1 and M_2 such that $M_1 \models \phi$ and $M_2 \models \neg\phi$.

   (b) Using the result about isomorphic Σ-structures mentioned above, show that both M_1 and M_2 must satisfy ϕ as well as $\neg\phi$. This is of course a contradiction; hence such a formula ϕ on the signature Σ cannot exist.