CS719 Mid-semester Exam #3

Max marks: 65 **Time: 2 hours**

- *The exam is open-book, open-notes and open-material-brought-to-exam-hall.*
- *Be brief, complete and stick to what has been asked. Unnecessarily lengthy solutions may be penalized.*
- *If you need to make any reasonable assumptions, state them clearly. Unreasonable assumptions run the risk of attracting penalty.*
- *If you need to use/cite results covered in class, you may simply cite the result, without going into a formal proof.*
- ***Do not copy from others or indulge in unfair means.***
  Students found indulging in such activities will be summarily awarded the FR grade.

1. *[5 + 5 + 5 + 5 marks]* Consider the first order logic sentence
   $\varphi \equiv \exists x \forall y \exists z\ ((x = y) \vee (P(x, y, z) \wedge \sim P(y, x, z)))$, where P is a ternary predicate.
   (a) Show that for every natural number $n \in \{1, 2, ...\}$, φ has a model of size n (the size of a model is the cardinality of its universe).
   (b) Let $M = (U^M, P^M)$ and $N = (U^N, P^N)$ be arbitrary models of φ, where U^M and U^N are countably infinite. Show that there exists a model $M \times N = (U^M \times U^N, P^{M\times N})$ of φ.
   (c) Can the result of part (ii) above be generalized to any first-order logic sentence ψ? Give brief justification for your answer.
   (d) Now consider an arbitrary first order logic sentence η that ***does not use the equality predicate***. A logician claims that if η has a model of finite size $n > 0$, then η must also have a model of size $n+1$. If you think the logician is correct, provide a proof of the claim. Otherwise, provide a counterexample, i.e. a sentence η and a natural number $n > 0$ such that η has a model of size n but no models of size $n + 1$.

2. *[10 + 10 marks]* Let $\Sigma = \{T, =\}$ be a vocabulary, where T is a ternary (arity 3) predicate. Every Σ-structure $M = (U^M, T^M)$ can be viewed as representing a directed graph $\mathcal{G}^M$ with red- and black-coloured edges as follows:
   - The elements of U^M are the vertices of $\mathcal{G}^M$.
   - For all $x, y, z \in U^M$, $T^M(x, y, z)$ is true iff there exists a black edge from x to z and a red edge from x to y in $\mathcal{G}^M$. Note that y and z may be identical, in which case x has both a black edge and a red edge to the same node (y).

Graphs representable by first order logic sentences over Σ have the special property that for every node x, either there is no edge going out of x, or there are both black edge(s) and red edge(s) going out of x. Let us call the class of directed graphs having the above property as *red-black graphs.*

(a) Given a red-black graph, a *source vertex* is a vertex that has no incoming edges, and a *non-cyclic path* is a directed path in which no two vertices are the same. Show that there does not exist any first order logic sentence φ on the vocabulary Σ such that $M \models \varphi$ if and only if $\mathcal{G}^M$ is a red-black graph with a non-cyclic path from a source vertex such that the edge colour changes exactly once along this path.

(b) Give a first order logic sentence ψ on the vocabulary Σ such that $M \models \psi$ if and only if $\mathcal{G}^M$ is a red-black graph such that the set of red edges forms a forest (collection of disjoint trees), but the set of black edges does not form a forest.

3. *[10 + 6 + 4 + 5 marks]* Consider the following propositional logic formulae on the set of propositional variables $\{p, q, r, s, t, u, v\}$.

- $\varphi \equiv (p \vee u) \wedge (p \vee v) \wedge (q \vee (u \wedge v)) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim q \vee \sim r) \wedge (\sim r \vee u) \wedge (\sim r \vee q)$.
- $\psi \equiv (\sim s \vee p) \wedge (\sim s \vee t) \wedge (\sim t \vee \sim u \vee q) \wedge ((p \wedge q) \vee (s \wedge u \wedge v))$

(a) Show using ground resolution that $\varphi \wedge \psi$ is unsatisfiable. In other words, $\varphi \rightarrow \sim \psi$.

(b) Compute three semantically distinct Craig interpolants ϕ_1, ϕ_2 and ϕ_3 for φ and $\sim \psi$ such that $\phi_1 \rightarrow \phi_2$ and $\phi_1 \wedge \sim \phi_3$ is satisfiable.

(c) Let η_1 and η_2 be propositional logic formulae such that $\eta_1 \wedge \eta_2$ is unsatisfiable. Show that there always exists a semantically unique weakest interpolant (one that is logically implied by all other interpolants) and a semantically strongest interpolant (one that logically implies all other interpolants) for η_1 and $\sim \eta_2$. Find the strongest and weakest interpolants for φ and $\sim \psi$.

(d) Is it always possible to find such strongest and weakest interpolants for a pair of first order logic sentences that are together unsatisfiable?