CS719 Practice Homework #1

- *Be brief, complete and stick to what has been asked.*
- ***Do not turn in your solutions. These problems are for you practice only.***

Problems on Propositional Logic:

1. We have learnt about clauses and conjunctive normal forms (CNF) of propositional logic formulae in class. A clause is said to be a *Horn clause* if it contains at most one positive (or unnegated) literal. A conjunction of Horn clauses is said to to be a *Horn formula.*

   As examples, $(\sim a \vee \sim b \vee c \vee d)$ is not a Horn clause since it has two unnegated literals. On the other hand, $(\sim a \vee \sim b \vee c)$ is a Horn clause, and $(\sim a \vee \sim b \vee c) \wedge (\sim c \vee a)$ is a Horn formula.

   It is known that a Horn formula can be checked for satisfiability in polynomial time (you are encouraged to find out how).

   Now consider the following variant of Horn clauses and formulae. A *relaxed Horn clause* is one that has two unnegated literals. The unnegated literals appearing in a relaxed Horn clause are said to be *paired.* A *relaxed Horn formula* φ is a CNF formula defined as follows:

   - Every clause is either a Horn clause or a relaxed Horn clause.
   - If u and v are paired literals in a relaxed Horn clause, then every clause C in φ satisfies the following:

     P1 C contains u iff it also contains v.

     P2 If $C \equiv (D \vee \sim u)$, then there exits a clause $C' \equiv (D \vee \sim v)$ in φ.

     P3 No clause contains both a literal and its negation.

   Show that the satisfiability of a relaxed Horn formula can be checked in polynomial time *[Hint: You need to first read up on how satisfiability of Horn formulas is checked in order to answer this question.]*

   Suppose properties P1 and P2 above are not satisfied by clauses of φ. If there are k relaxed Horn clauses in φ, show that satisfiability of φ can be checked in time that is polynomial in n (size of the formula) and exponential in k.

2. Suppose we are given two (potentially infinite) sets of propositional logic formulae $\mathcal{F}$ and $\mathcal{G}$ on the same set, X, of propositional variables.

   - We say that $\mathcal{F} \equiv \mathcal{G}$ iff every satisfying assignment μ of $\mathcal{F}$ is also a satisfying assignment of $\mathcal{G}$ and vice versa.
   - We say that $\mathcal{F} \simeq_1 \mathcal{G}$ iff for every $\varphi \in \mathcal{F}$, $\mathcal{G} \vdash \varphi$, and for every $\psi \in \mathcal{G}$, $\mathcal{F} \vdash \psi$.
   - We say that $\mathcal{F} \simeq_2 \mathcal{G}$ iff for every $\varphi \in \mathcal{F}$, there exists some $\psi \in \mathcal{G}$ such that $\psi \vdash \varphi$, and for every $\psi \in \mathcal{G}$, there exists some $\varphi \in \mathcal{F}$ such that $\varphi \vdash \psi$.

   Are the three relations $\equiv, \simeq_1, \simeq_2$ equivalent? If not, which of them is the strongest and which is the weakest?

3. A set $\mathcal{S}$ of propositional logic formulae is said to form a *minimal unsatisfiable core* if $\mathcal{S}$ is unsatisfiable, but every proper subset of $\mathcal{S}$ is satisfiable.

   (a) Given an unsatisfiable set $\mathcal{S}$ of propositional logic formulae, does there always exist a *unique* subset of $\mathcal{S}$ that forms a minimal unsatisfiable core? If so, give a proof. Otherwise, give a counterexample.

   (b) Show that there exists a minimal unsatisfiable core of n propositional logic formulae, for every $n > 0$.

   (c) The compactness theorem tells us that if $\mathcal{S}$ is an unsatisfiable set of propositional logic formulae, there must exist a finite subset of $\mathcal{S}$ that is unsatisfiable. Does this contradict the statement of the previous subquestion? If not, why?

4. Let φ and ψ be propositional logic formulae such that $\models \varphi \rightarrow \psi$ (i.e. $\varphi \rightarrow \psi$ is a tautology). Let $Var(\varphi)$ and $Var(\psi)$ denote the set of propositional variables in φ and ψ respectively. Show that there exists a propositional logic formula ζ with $Var(\zeta) = Var(\varphi) \cap Var(\psi)$ such that $\models \varphi \rightarrow \zeta$ and $\models \zeta \rightarrow \psi$. This result is also known as *Craig's interpolation theorem* as applied to propositional logic.

5. We have studied ground resolution for propositional logic in class.

   (a) Use ground resolution to establish the following:

      i. $\sim A$ is a consequence of $(A \rightarrow B) \wedge (A \rightarrow \sim B)$.

      ii. The set of formulae $\{(A \rightarrow B), (B \rightarrow C), (C \rightarrow \sim A)\}$ is satisfiable.

   (b) Consider the following pseudo-code for ground-resolution:

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Input: A propositional logic formula F in CNF.

Res_0 = {C | C is a clause of F};
n = 0;
do {
 Res_{n+1} = Res_n Union {R | R is a resolvent of two clauses in Res_n};
 n = n+1;
} until ((empty clause in Res_n) or (Res_n = Res_{n-1}))
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   If F has k propositional variables, what is the maximum number of times the above loop can iterate?

   What is the maximum number of new resolvents that can be added in the n^{th} iteration of the loop?