CS719 Practice Homework #2

- *Be brief, complete and stick to what has been asked.*
- ***Do not turn in your solutions. These problems are for you practice only.***

Problems on First-order Logic and Structures:
In class, we studied about *interpretations* of first-order logic formulae. An interpretation is also sometimes called a *structure*; we will therefore use structures and interpretations interchangeably in the following questions.
All formulae in the following questions (except the second question below) are assumed to be free of free variables, i.e. all variables are bound. Such formulae are also sometimes called *closed formulae* or *sentences*, and play a very important role in the study of logic.
Recall further that an interpretation (structure) $\mathcal{M}$ is called a *model* of a sentence φ if $\mathcal{V}^{\mathcal{M}}(\varphi) = \mathsf{true}$. This is also commonly written as $\mathcal{M} \models \varphi$. We write $\models \varphi$ to denote that φ is valid, i.e., evaluates to true in all Σ-structures, where Σ is the vocabulary of φ.

1. Give an example of a first-order logic sentence φ such that every model of φ necessarily has an infinite universe. You are allowed to have the equality predicate, i.e "=", as part of your vocabulary. Note that every model $\mathcal{M}$ must interpret "=" as follows: For all $a, b \in U_{\mathcal{M}}$, $=^{\mathcal{M}} (a, b)$ is true iff a and b are the same elements. You must try to minimize the number of predicate and function symbols (other than "=") that you need in the vocabulary.

2. Let $\Sigma = \{P, a, b\}$ be a signature, where P is a unary predicate, and a and b are nullary functions (or constants). Let $\varphi(x)$ be the first-order logic formula $P(x) \rightarrow (P(a) \wedge P(b))$ with signature Σ. Show that there is no term t that is free for x in φ such that $\models \varphi(t)$. Show also that $\models \exists x\, \varphi(x)$. Is there a contradiction in the above two statements? Explain why.

3. Let Σ_{gp} be the vocabulary $\{+, 0, =\}$, where "+" is a binary function symbol, "=" is the binary equality predicate, and "0" is a nullary function symbol (or constant). You must resist the temptation of interpretting "+" as arithmetic addition, and "0" as the additive identity in arithmetic. In fact, different Σ_{gp}-structures may assign different interpretations to "+" and "0". However, the interpretation of "=" must stay the same across all structures. For convenience of notation and of reading, we will write $x + y$ instead of $+(x, y)$, and $x = y$ instead of $= (x, y)$ in the following.

   Consider the following first-order logic sentences over the vocabulary Σ_{gp}.

   - $\varphi_1 \equiv \forall x \forall y \forall z\, (x + (y + z) = (x + y) + z)$
   - $\varphi_2 \equiv \forall x\, ((x + 0 = x) \wedge (0 + x = x))$
   - $\varphi_3 \equiv \forall x\, (\exists y(x + y = 0) \wedge \exists z(z + x = 0))$

   Let $\varphi \equiv \varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

   - Show that φ is satisfiable by giving a model.
   - Show that φ is not valid by giving a Σ-structure $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi$.
   - Show that $\forall x \forall y\, ((x + y) = (y + x))$ is not logically implied by φ.

- Show that the conjunction of any two of φ_1, φ_2 and φ_3 does not logically imply φ.

4. Let Σ_G be the vocabulary $\{E, =\}$ where E is a binary predicate, and "=" is the usual equality predicate. Every Σ_G structure $\mathcal{M} = (U_\mathcal{M}, E^\mathcal{M})$ can be interpretted as a directed graph, where the elements of $U_\mathcal{M}$ are the nodes, and for all $a, b \in U_\mathcal{M}$, there exists an edge from a to b iff $E^\mathcal{M}(a, b)$ is true. Similarly, every directed graph can be interpretted as a Σ_G structure. Thus, every first-order logic sentence φ on the vocabulary Σ_G can be thought of as describing a set of directed graphs, i.e. the set of graphs that serve as models of φ.

   (a) Give a formula φ_1 on the vocabulary Σ_G such that a graph $\mathcal{G}$ is a model of φ_1 iff G has no cliques of size > 5.

   (b) Give a formula φ_2 on Σ_G such that there are arbitrarily large finite graphs that serve as models of φ_2, and any finite graph that is a model of φ_2 necessarily has an even number of nodes.

   (c) Given any finite graph $\mathcal{G}$, show that there exists a first order logic sentence $\varphi_\mathcal{G}$ on the vocabulary Σ_G such thate every model of $\varphi_\mathcal{G}$ is necessarily isomorphic to $\mathcal{G}$. Can this result be extended to graphs with countably infinite nodes?

   (d) Suppose you are told that the Compactness Theorem holds for first-order logic as well[1], i.e. given a set of first-order logic sentences that is unsatisfiable, there must exist a finite subset of the set that is unsatisfiable. Can you use this to show that there cannot exist any first-order logic formula φ on the signature Σ_G such that a graph satisfies φ iff it does not contain a cycle? This illustrates one of the many useful applications of Compactness Theorem, whereby we can conclude that something cannot be expressed in first-order logic.

[1]We will of course prove this in class