CS719 Practice Homework #3

- *Be brief, complete and stick to what has been asked.*
- ***Do not turn in your solutions. These problems are for your practice only.***

Problems on Propositional and First-order Logics:

1. We are given an empty tumbler of volume V and a set of n glasses $\{g_1, \ldots g_n\}$, each filled with water. Each glass g_i has a volume $v_i > 0$. The volumes of different glasses are not necessarily identical. It is known that $\Sigma_{i=1}^{n} v_i > V$. Therefore if all the glasses were emptied into the tumbler, the tumbler would definitely overflow. Our goal is to fill up the tumbler at least upto volume $D\,(D \leq V)$ without overflowing. However, we have the restriction that a glass can either be completely emptied into the tumbler or not a drop of its water can be poured into the tumbler. In other words, you can't pour part of the water in a glass into the tumbler. You may assume that V, D and each v_i are integers.

   We wish to use propositional logic to find out if it is indeed possible to fill up the tumbler at least upto volume D without overflowing, under the constraints mentioned above. In other words, we want to construct a propositional logic formula on a suitably defined set of propositions, such that the formula is satisfiable if and only if the tumbler can be filled at least upto volume D without overflowing.

   (a) Indicate what set of propositions you would use to solve this problem. You must indicate what the truth value of each such proposition means in the context of the original problem. The total number of propositions must be polynomial in n and D (recall D is an integer). Any solution with more propositions is not acceptable.

   (b) Indicate how you will construct a propositional logic formula to capture the constraints in the above problem. Your formula must be in the form of a conjunction of different subformulae, where each subformula captures some of the constraints in the above problem.

2. Let p, q be propositions, and ϕ_1 and ϕ_2 be propositional logic formulae on p, q.

   (a) Consider the following definitions for ϕ_1 and ϕ_2
   - $\phi_1 = (p \rightarrow\sim \phi_2)$
   - $\phi_2 = (q \rightarrow\sim \phi_1)$

   Show that there are exactly two pairs of propositional logic formulae (ϕ_1, ϕ_2) which satisfy the above definitions. Also, justify your answer.

   (b) If the definition of ϕ_1 above was changed to $\phi_1 = (p \rightarrow \phi_2)$, and the definition of ϕ_2 was left unchanged, is it possible to find propositional logic formulae on propositions p and q that satisfy the modified definitions? If your answer is in the negative, you must explain why. Otherwise, you must give the propositional logic formulae ϕ_1 and ϕ_2 in terms of p and q, and provide justification for your answer.

3. Consider the first order logic sentence

$$\phi = \forall x \exists y\, (P(x, y) \vee \forall z \exists w\, (Q(z, w) \rightarrow\ R(x, z)))$$

   (a) Convert ϕ to Skolem Normal Form (SNF). You must use Skolem functions with as small arities (number of arguments) as possible. Remember that in SNF, the matrix must be in CNF.

   (b) A model $\mathcal{M}$ for ϕ consists of a universe $\mathcal{D}$ and an interpretation of predicates P, Q, R over $\mathcal{D}$, for which ϕ evaluates to True. Give a model $\mathcal{M}_1$ for ϕ and a model $\mathcal{M}_2$ for $\sim \phi$.

   (c) Let ψ denote the SNF of ϕ as obtained in part (a). We know that ψ and ϕ are equisatisfiable, and that a model for ψ can be obtained from a model for ϕ and vice versa.

      i. Given the model $\mathcal{M}_1$ for ϕ in part (b) above, what is the corresponding model that can be derived for ψ? Note that since ψ has Skolem functions, the model for ψ must have interpretations of the Skolem functions in addition to interpretations of the predicates P, Q, R.

      ii. Are $\sim \phi$ and $\sim \psi$ equivalid? Give justification for your answer.

4. Consider the first order logic sentence $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$, where

$$\begin{aligned}
\phi_1 &= \forall x\, (f(g(x)) = g(x)), \\
\phi_2 &= \forall x\, (g(f(x)) = f(x)), \\
\phi_3 &= \forall x \exists y\, ((x = g(y)) \vee (x = f(y)))
\end{aligned}$$

   (a) Let $\mathcal{H}$ be the Herbrand Universe for the sentence ϕ (of course, obtained after Skolemizing ϕ). Let $M_{\mathcal{H}}$ be a Herbrand structure with the universe being $\mathcal{H}$, and with the obvious interpretations of functions f and g. If $M_{\mathcal{H}} \models \phi$, how many *distinct elements* can be present in the universe of $M_{\mathcal{H}}$. Give justification for your answer.

   (b) Prove using the proof system of first order logic with equality that $\phi_1, \phi_2, \phi_3\ \vdash\ (x = f(x))$.

   (c) Let $\phi_4\ =\ \exists x\ \sim (f(x) = g(x))$. Show using Herbrand's Theorem that $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$ is unsatisfiable. Your solution for this part must not use results obtained in previous parts.

5. Once upon a time, there was a logician who, by some strange stroke of fate, ended up ruling a land (such things aren't common in recent times!) Being a logician, the ruler was interested in finding out whether the prime minister, who was entrusted with key responsibilities, was logically consistent in thinking. So one evening, the prime minister (PM) was summoned and was asked to respond in "Yes/No" to questions that the ruler (R) would pose.

   The following short question-answer session ensued:

   R: Is there a happy person in my empire who knows somebody who in turn knows an unhappy person?

   PM: No

   R: Is there a happy person in my empire who is not known to even one other happy person?

   PM: No

   R: Is there a happy person in my empire who knows another person who is unhappy?

   PM: Yes

At this point, the logician-turned-ruler remarked that the prime minister is being logically inconsistent and recommended that a crash course in first order logic be given to the prime minister. As the first assignment, the minister was asked to use to prove that the sequence of answers given above is logically inconsistent. We must help the minister in this noble effort.

We will use a unary predicate $H(x)$ that evaluates to true iff x is happy, and a binary predicate $K(x, y)$ that evaluates to true iff x knows y, in addition to the usual equality predicate (if needed). No other predicates or functions must be used. Note that K is not necessarily a reflexive, symmetric or transitive relation, and we must not make any such assumptions.

(a) Express the information provied by each of the above question-answer pairs as a formula in first order logic. Thus, you should obtain three formulae ϕ_1, ϕ_2, ϕ_3 using the predicates H and K (possibly in addition to the equality predicate), and using no other predicates or functions.

(b) Show using Herbrand's Theorem that $\phi_1 \wedge \phi_2 \wedge \phi_3$ is unsatisfiable.

(c) Can you also show the above unsatisfiability using resolution?

6. Let $\phi_{K,N}$ be a first order logic sentence with signature $\{P, =\}$ Suppose further that P is a unary predicate. A model $\mathcal{M}$ for $\phi_{K,N}$ consists of a set of elements $D_{\mathcal{M}}$, called the universe, and an interpretation $P_{\mathcal{M}} : D_{\mathcal{M}} \to \{\mathsf{true}, \mathsf{false}\}$ of the predicate P. Therefore a model $\mathcal{M}$ for $\phi_{K,N}$ can be represented as a pair $(D_{\mathcal{M}}, P_{\mathcal{M}})$. You are told the following additional facts about $\phi_{K,N}$ and its models.

- In every model $\mathcal{M}$ such that $\mathcal{M} \models \phi_{K,N}$, the universe $D_{\mathcal{M}}$ contains *at most* K distinct elements, and the interpretation $P_{\mathcal{M}}$ of P evaluates to true on *at least* N distinct elements of $D_{\mathcal{M}}$.
- Any model $\mathcal{M}$ in which $D_{\mathcal{M}}$ contains *at most* K distinct elements and the interpretation $P_{\mathcal{M}}$ of P evaluates to true on *at least* N distinct elements of $D_{\mathcal{M}}$, satisfies $\mathcal{M} \models \phi_{K,N}$.

(a) Give a first order logic sentence $\phi_{K,N}$ satisfying the above conditions. You may use the notation $\exists x_1 \exists x_2 \ldots \exists x_r$ or $\forall y_1 \forall y_2 ... \forall y_r$ to denote a sequence of r quantifications of the same type ($\exists$ or $\forall$).

(b) Show using Herbrand's Theorem that $\phi_{2,3}$ is unsatisfiable.

(c) Can you use resolution to show that $\phi_{3,3} \to \forall x\, P(x)$ is a valid sentence?

In the last two subquestions, $\phi_{2,3}$ and $\phi_{3,3}$ must be obtained by substituting appropriate values for K and N in your answer to the first subquestion.

7. In this question, we wish to express properties of directed graphs using first order logic. Consider a first order logic formula ϕ containing a single binary predicate E in addition to the equality predicate. A model $\mathcal{M}$ for ϕ consists of a universe $D_{\mathcal{M}}$ and an interpretation $E_{\mathcal{M}} : D_{\mathcal{M}} \times D_{\mathcal{M}} \to \{\mathsf{true}, \mathsf{false}\}$ of the predicate E. Such a model $\mathcal{M}$ can also be viewed as a directed graph $G_{\mathcal{M}}$, where the elements of $D_{\mathcal{M}}$ are the vertices of the graph, and a directed edge exists from a to b if and only if $E_{\mathcal{M}}(a, b)$ is true. For purposes of this question, we will assume that $D_{\mathcal{M}}$ is finite.

In each of the following cases, you are required to either show that the class of graphs $G_{\mathcal{M}}$ described below cannot be described by any first order logic sentence (think Compactness Theorem) or provide a first order logic sentence such that $\mathcal{M} \models \phi$ if and only if

(a) $G_{\mathcal{M}}$ has exactly two maximal disconnected cliques.

(b) $G_{\mathcal{M}}$ has exactly two maximal cliques, not necessarily disconnected.

(c) $G_{\mathcal{M}}$ is a tree with at most 4 nodes and with a unique root.

(d) $G_{\mathcal{M}}$ is a forest **where each tree in the forest has at most** 4 **nodes**. Note that a forest is a collection of one or more trees, each with a unique root.

(e) $G_{\mathcal{M}}$ is a collection of one or more simple cycles. Note that a simple cycle is a graph in which (i) it is possible to start from any node and follow directed edges to return to the same node, and (ii) the *only way* to do the above (without visiting the starting node in between) is by visiting all other nodes on the simple cycle exactly once.

(f) There is at least one infinite path (containing repeated vertices of course) starting from every vertex in $G_{\mathcal{M}}$.

8. In this question, we wish to state certain properties of natural numbers in first order logic. You may use the interpretted predicates $<$ and $=$, and interpretted functions $*$ and $+$ on natural numbers with the usual interpretation. You may also use $one()$ as an interpretted nullary function that returns the value 1.

   Give first order logic sentences expressing the following properties of natural numbers.

   (a) There are natural numbers that cannot be expressed as one natural number raised to the power of another natural number distinct from 1.

   For this question, you are allowed to use additional ***uninterpretted*** *function symbols (other than $*$ and $+$) in your answer φ. However, every model that satisfies φ and interprets $+$, $*$, $<$ and $=$ as described above, must necessarily assert the above property of natural numbers. In other words, it should not be possible to find a model of φ unless the above property holds for natural numbers. Note the stress on additional functions being* ***uninterpretted*** *in the above requirement. Specifically, you are not allowed to use an interpretted exponentiation function directly.*

   (b) There are natural numbers that cannot be expressed as the product of distinct natural numbers, none of which is 1.

   (c) There are infinitely many natural numbers that have only one way of factorizing them as the product of two natural numbers.

9. Consider the following sentence in first order logic:
   $\phi_1 : \ \forall x \, \exists y \, \exists z \ (P(x,y) \wedge (P(x,z) \rightarrow \sim (z = y)))$

   Let ϕ_2 be an unspecified first order logic sentence in SNF that has k nullary functions, and has no function of arity greater than 0. Note that ϕ_2 can contain instances of predicate P.

   A logician now claims that she can check the satisfiability of $\phi_1 \wedge \phi_2$ for any ϕ_2 satisfying the above conditions, by considering models having finite universe of cardinality at mosk k.

   If you think the logician is correct, indicate why models with universe of cardinality k suffice to check satisfiability. Otherwise, give justification why the logician is incorrect.

10. In this question, we will talk about models of first order logic formulae with only one binary predicate symbol L other than $=$. Let $\mathcal{M}_1 = (A_1, L_1)$ and $\mathcal{M}_2 = (A_2, L_2)$ be two models, where A_i denotes a universe of elements and $L_i \subseteq A_i \times A_i$ is an interpretation of the predicate L. Models $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be *isomorphic* iff there exists a *bijection* $h : A_1 \to A_2$ such that $L_2(h(x_1), h(y_1)) = L_1(x_1, y_1)$ for all $x_1, y_1 \in A_1$ (and conversely, $L_1(h^{-1}(a_1), h^{-1}(a_2)) = L_2(a_1, a_2)$ for all $a_1, a_2 \in A_2$).

    (a) Give a first order logic sentence ϕ such that every countable model $\mathcal{M}$ of ϕ is isomorphic to $(\mathbf{N}, <)$, i.e. the set of natural numbers with the usual less-than binary predicate.

    (b) Give a first order logic sentence ψ such that every countable model of ψ is isomorphic to a rooted countably infinite binary tree in which every path is of infinite length. Clearly, such a binary tree qualifies to be a model $\mathcal{M}_1 = (A_1, L_1)$, where A_1 is the set of nodes of the tree and $L_1(x, y)$ is true iff y is a child of x.

    (c) Is it possible to have a first order logic sentence ϕ such that every model of ϕ is isomorphic to $(\Re, <)$, i.e. the set of real numbers with the usual less-than binary predicate? You must justify your answer.

    (d) Let ϕ be a sentence in first order logic and let $\mathcal{M}_1 = (A_1, L_1)$ and $\mathcal{M}_2 = (A_2, L_2)$ be two isomorphic models of ϕ with $A_1 \cap A_2 = \emptyset$. Is it then necessarily true that ϕ also has a model $\mathcal{M}_3 = (A_3, L_3)$ such that $\mathcal{M}_1$ and $\mathcal{M}_2$ are submodels? In other words, can we have $\mathcal{M}_3 \models \phi$, with $A_3 = A_1 \cup A_2$ and $L_3 \cap (A_1 \times A_1) = L_1$ and $L_3 \cap (A_2 \times A_2) = L_2$? You must justify your answer.

11. In this question, we wish to investigate the expressiveness of first order logic in comparing cardinalities of sets. Towards this objective, let P and Q be unary predicates.

    (a) Is it possible to have a first order logic sentence ϕ over a signature Σ that includes P, Q and $=$ (and perhaps other functions and predicates) such that (i) for every Σ-structure M that satisfies ϕ, the cardinality of the set of elements in the universe of M for which P evaluates to true is equal to the cardinality of the set of elements for which Q evaluates to true, and (ii) every structure M' (on a signature Σ' that contains P and Q) for which the above cardinalities are equal can be extended to a structure M'' by keeping the interpretations of P and Q unchanged, and perhaps by adding interpretations of additional funcitons and predicates, such that $M'' \models \phi$?

    (b) Is it possible to have a first order logic sentence ϕ such that (i) for every model satisfying ϕ, there are only a finite number of elements for which P evaluates to true, and (ii) every model which has finitely many elements for which P evaluates to true, also satisfies ϕ?

    In both sub-questions above, you must either provide the first order logic sentence along with justification for why the sentence satisfies the required conditions, or you must prove that such a sentence cannot exist.